Solutions to the Exercises of Lecture 14

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Exercise 1 (and "a little bit" more)

Let $I \subseteq \mathbb{R}$ be an interval, $f \in C(I)$. For every $\alpha \in [0,1]$ we define the Hölder seminorm

$$[f]_\alpha := \sup_{\substack{x,y \in I \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

If $\alpha \in (n, n+1]$ for some $n \in \mathbb{N}$, and if $f \in C^n(I)$, then we put

$$[f]_\alpha := [f^{(n)}]_{\alpha - n},$$

so that for integer $\alpha \geq 1$ and $f \in C^\alpha(I)$

$$[f]_\alpha = \|f^{(\alpha)}\|_\infty.$$

For $\alpha \in [n, n+1)$ $(n \in \mathbb{N}_0)$ we define the space

$$C_b^\alpha(I) := \Big\{ f \in C^n(I) ;\ \sum_{j=0}^n \|f^{(j)}\|_\infty + [f]_\alpha < \infty \Big\}$$

and similarly, for $\alpha \in (n, n+1]$ $(n \in \mathbb{N}_0)$, we define the space

$$C_b^{\alpha-}(I) := \Big\{ f \in C^n(I) ;\ \sum_{j=0}^n \|f^{(j)}\|_\infty + [f]_\alpha < \infty \Big\}.$$

Both spaces are equipped with obvious norms. Note that

$$\begin{aligned} &C_b^\alpha(I) = C_b^{\alpha-}(I) \text{ whenever } \alpha \notin \mathbb{N}_0, \\ &C_b^\alpha(I) \text{ is a closed subspace of } C_b^{\alpha-}(I) \text{ whenever } \alpha \in \mathbb{N}, \\ &\qquad \text{the corresponding norms are equivalent on } C_b^\alpha(I), \text{ and} \\ &C_b^{\beta-}(I) \subseteq C^\alpha(I) \text{ whenever } 0 \leq \alpha < \beta. \end{aligned}$$

Theorem 1.1. *For every $0 \leq \alpha < \beta$ there exists a constant $C \geq 0$ such that for every interval $I \subseteq \mathbb{R}$, $0 < d < \text{length}(I)$, $\gamma \in [\alpha, \beta]$, and $f \in C_b^{\beta-}(I)$*

$$[f]_\gamma \leq C\left(\frac{1}{d^\gamma}\,[f]_\alpha + [f]_\alpha^\theta\,[f]_\beta^{1-\theta}\right),$$

where $\theta = \frac{\beta-\gamma}{\beta-\alpha}$ (so that $\gamma = \theta\alpha + (1-\theta)\beta$).

We prove this assertion first for α, $\beta \in [0,1]$, then for $\alpha \in [0,1)$ and $\beta \in (1,2]$, and finally by an induction argument for general α, β. The first two cases, restricted to the case of the interval $I = [0,1]$, are stated in a separate lemma. By choosing $\alpha = 0$, $\beta = 2$ and $\gamma = 1$ in assertion (b) of the following lemma, one obtains an answer to Exercise 14.1.

Lemma 1.2. (a) *For every $0 \leq \alpha < \beta \leq 1$, $\gamma \in [\alpha, \beta]$ and $f \in C^{\beta-}([0,1])$*

$$[f]_\gamma \leq [f]_\alpha^\theta\,[f]_\beta^{1-\theta},$$

where $\theta = \frac{\beta-\gamma}{\beta-\alpha}$ (so that $\gamma = \theta\alpha + (1-\theta)\beta$).

(b) *For every $\alpha \in [0,1)$, $\beta \in (1,2]$, $\gamma \in [\alpha, \beta]$ and $f \in C^{\beta-}([0,1])$*

$$[f]_\gamma \leq 2[f]_\alpha + 2\,[f]_\alpha^\theta\,[f]_\beta^{1-\theta} \quad \text{if } \gamma \leq 1, \text{ and}$$
$$[f]_\gamma \leq 2[f]_\alpha + 4\,[f]_\alpha^\theta\,[f]_\beta^{1-\theta} \quad \text{if } \gamma > 1,$$

where $\theta = \frac{\beta-\gamma}{\beta-\alpha}$ (so that $\gamma = \theta\alpha + (1-\theta)\beta$).

Proof. (a) One has

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|^\gamma} &= \frac{|f(x)-f(y)|^\theta\,|f(x)-f(y)|^{1-\theta}}{|x-y|^{\theta\alpha}\,|x-y|^{(1-\theta)\beta}} \\ &\leq [f]_\alpha^\theta\,[f]_\beta^{1-\theta},\end{aligned}$$

whence the claim.

(b) *First case:* $\gamma = 1$ (includes the case of Exercise 14.1!). Then $\theta = \frac{\beta-1}{\beta-\alpha}$. For every x, $x+h \in [0,1]$ the mean value theorem implies

$$f(x+h) - f(x) = f'(x+th)h - f'(x)h + f'(x)h$$

for some $t \in [0,1]$, and therefore

$$\begin{aligned}|f'(x)| &\leq \frac{|f(x+h)-f(x)|}{|h|^\alpha}\,\frac{1}{|h|^{1-\alpha}} + \frac{f'(x+th)-f'(x)}{|th|^{\beta-1}}\,|th|^{\beta-1} \\ &\leq [f]_\alpha\,\frac{1}{|h|^{1-\alpha}} + [f']_{\beta-1}\,|h|^{\beta-1}.\end{aligned}$$

Since h was arbitrary, but since in general $|h| \le \frac{1}{2}$ (the maximal possible choice if $x = \frac{1}{2}$), we find

$$[f]_1 = \|f'\|_\infty \le \inf_{0<r\le\frac{1}{2}} \Big([f]_\alpha \, \frac{1}{r^{1-\alpha}} + [f]_\beta \, r^{\beta-1} \Big)$$

By choosing $r = \big(\frac{[f]_\alpha}{[f]_\beta}\big)^{\frac{1}{\beta-\alpha}} \wedge \frac{1}{2}$, we obtain

$$\begin{aligned} [f]_1 &\le ([f]_\alpha^\theta \, [f]_\beta^{1-\theta}) \vee (2^{1-\alpha} \, [f]_\alpha) + [f]_\alpha^\theta \, [f]_\beta^{1-\theta} \\ &\le 2[f]_\alpha + 2 \, [f]_\alpha^\theta \, [f]_\beta^{1-\theta}. \end{aligned}$$

Second case: $\gamma < 1$. By using assertion (a) and the previous case,

$$\begin{aligned} [f]_\gamma &\le [f]_\alpha^{\frac{1-\gamma}{1-\alpha}} \, [f]_1^{\frac{\gamma-\alpha}{1-\alpha}} \\ &\le [f]_\alpha^{\frac{1-\gamma}{1-\alpha}} \big(2 \, [f]_\alpha + 2 \, [f]_\alpha^{\frac{\beta-1}{\beta-\alpha}} \, [f]_\beta^{\frac{1-\alpha}{\beta-\alpha}} \big)^{\frac{\gamma-\alpha}{1-\alpha}} \\ &\le 2 \, [f]_\alpha^{\frac{1-\gamma}{1-\alpha}} \big([f]_\alpha^{\frac{\gamma-\alpha}{1-\alpha}} + [f]_\alpha^{\frac{\beta-1}{\beta-\alpha}\,\frac{\gamma-\alpha}{1-\alpha}} \, [f]_\beta^{\frac{\gamma-\alpha}{\beta-\alpha}} \big) \\ &= 2 \, [f]_\alpha + 2 \, [f]_\alpha^\theta \, [f]_\beta^{1-\theta}. \end{aligned}$$

Third case: $\gamma > 1$. By using assertion (a) (applied to f') and the first case,

$$\begin{aligned} [f]_\gamma &\le [f]_1^{\frac{\beta-\gamma}{\beta-1}} \, [f]_\beta^{\frac{\gamma-1}{\beta-1}} \\ &\le \big(2 \, [f]_\alpha + 2 \, [f]_\alpha^{\frac{\beta-1}{\beta-\alpha}} \, [f]_\beta^{\frac{1-\alpha}{\beta-\alpha}} \big)^{\frac{\beta-\gamma}{\beta-1}} \, [f]_\beta^{\frac{\gamma-1}{\beta-1}} \\ &\le 2 \, \big([f]_\alpha^{\frac{\beta-\gamma}{\beta-1}} + [f]_\alpha^{\frac{\beta-\gamma}{\beta-\alpha}} \, [f]_\beta^{\frac{1-\alpha}{\beta-\alpha}\,\frac{\beta-\gamma}{\beta-1}} \big) \, [f]_\beta^{\frac{\gamma-1}{\beta-1}} \\ &= 2[f]_\alpha^{\frac{\beta-\gamma}{\beta-1}} \, [f]_\beta^{\frac{\gamma-1}{\beta-1}} + 2 \, [f]_\alpha^\theta \, [f]_\beta^{1-\theta}. \end{aligned}$$

Applying now Young's inequality $ab \le \frac{1}{p} a^p + \frac{1}{q} b^q$ with $p = \frac{(\gamma-\alpha)(\beta-1)}{(\beta-\gamma)(1-\alpha)}$ and $q = \frac{(\gamma-\alpha)(\beta-1)}{(\beta-\alpha)(\gamma-1)}$, we have

$$\begin{aligned} [f]_\alpha^{\frac{\beta-\gamma}{\beta-1}} \, [f]_\beta^{\frac{\gamma-1}{\beta-1}} &= [f]_\alpha^{\frac{\beta-\gamma}{\beta-1} - \frac{\theta}{q}} \big([f]_\alpha^{\frac{\theta}{q}} \, [f]_\beta^{\frac{\gamma-1}{\beta-1}} \big) \\ &\le [f]_\alpha + [f]_\alpha^\theta \, [f]_\beta^{1-\theta}. \end{aligned}$$

Combining this inequality with the above inequality yields the claim. □

Proof of Theorem 1.1. Preliminary remark. Note that if the interpolation inequality is true for all $0 \le \alpha < \gamma < \beta \le n$ (for some $n \in \mathbb{N}$), then it is true for all $k \le \alpha < \gamma < \beta \le n + k$ with $k \in \mathbb{N}_0$. Simply apply the first case to $f^{(k)}$. In the following we prove the assertion only for the unit interval $[0, 1]$. The case of an arbitrary interval follows by a rescaling argument and noting that

$$[f(d\cdot)]_\gamma = d^\gamma \, [f]_\gamma.$$

So assume that $I = [0,1]$. We proceed by induction. We know from Lemma 1.2 that the assertion is true for every $0 \le \alpha < \beta \le 2$. Now assume that the assertion is true for every $0 \le \alpha < \beta \le n$ for some $n \in \mathbb{N}$, $n \ge 2$. We show that the assertion is true for every $0 \le \alpha < \beta \le n+1$. The cases $0 \le \alpha < \beta \le n$ and $1 \le \alpha < \beta \le n+1$ follow directly from the induction hypothesis. Therefore it suffices to consider the case $\alpha \in [0,1)$ and $\beta \in (n, n+1]$. Let $\gamma \in [\alpha, \beta]$ and $f \in C^{\beta-}([0,1])$. As in the proof of Lemma 1.2 (b), we consider three cases.

First case: $\gamma = 1$. In this case we estimate

$$\begin{aligned}
[f]_1 &\le C\left([f]_\alpha + [f]_\alpha^{\frac{n-1}{n-\alpha}} [f]_n^{\frac{1-\alpha}{n-\alpha}}\right) \\
&\le C\left([f]_\alpha + [f]_\alpha^{\frac{n-1}{n-\alpha}} C^{\frac{1-\alpha}{n-\alpha}} \left([f]_1 + [f]_1^{\frac{\beta-n}{\beta-1}} [f]_\beta^{\frac{n-1}{\beta-1}}\right)^{\frac{1-\alpha}{n-\alpha}}\right) \\
&\le C\,[f]_\alpha + C^{1+\frac{1-\alpha}{n-\alpha}} [f]_\alpha^{\frac{n-1}{n-\alpha}} [f]_1^{\frac{1-\alpha}{n-\alpha}} + \\
&\quad + C^{1+\frac{1-\alpha}{n-\alpha}} [f]_\alpha^{\frac{n-1}{n-\alpha}} [f]_1^{\frac{\beta-n}{\beta-1}\frac{1-\alpha}{n-\alpha}} [f]_\beta^{\frac{n-1}{\beta-1}\frac{1-\alpha}{n-\alpha}}.
\end{aligned}$$

Now we use Young's inequality in the form $ab \le \frac{1}{p}(\frac{4}{q})^{\frac{p}{q}} a^p + \frac{1}{4} b^q$ (first with the pair $p = \frac{n-\alpha}{n-1}$ and $q = \frac{n-\alpha}{1-\alpha}$, and second with the pair $p = \frac{(\beta-1)(n-\alpha)}{(\beta-\alpha)(n-1)}$ and $q = \frac{(\beta-1)(n-\alpha)}{(\beta-n)(1-\alpha)}$) in order to estimate the second and the third term. We obtain

$$\begin{aligned}
[f]_1 &\le \left(C + C^{\frac{n+1-2\alpha}{n-1}} \left(\frac{4(1-\alpha)}{n-\alpha}\right)^{\frac{1-\alpha}{n-1}}\right) [f]_\alpha + \frac{1}{2}\,[f]_1 + \\
&\quad + C^{\frac{(n+1-2\alpha)(\beta-1)}{(n-1)(\beta-\alpha)}} \left(\frac{4(\beta-n)(1-\alpha)}{(\beta-1)(n-\alpha)}\right)^{\frac{(\beta-n)(1-\alpha)}{(\beta-\alpha)(n-1)}} [f]_\alpha^{\frac{\beta-1}{\beta-\alpha}} [f]_\beta^{\frac{1-\alpha}{\beta-\alpha}},
\end{aligned}$$

and from here follows the claim for $\gamma = 1$.

Second case: $\gamma < 1$. We estimate by using Lemma 1.2 (a) and the first case

$$\begin{aligned}
[f]_\gamma &\le [f]_\alpha^{\frac{1-\gamma}{1-\alpha}} [f]_1^{\frac{\gamma-\alpha}{1-\alpha}} \\
&\le [f]_\alpha^{\frac{1-\gamma}{1-\alpha}} C^{\frac{\gamma-\alpha}{1-\alpha}} \left([f]_\alpha + [f]_\alpha^{\frac{\beta-1}{\beta-\alpha}} [f]_\beta^{\frac{1-\alpha}{\beta-\alpha}}\right)^{\frac{\gamma-\alpha}{1-\alpha}} \\
&\le C\left([f]_\alpha + [f]_\alpha^\theta [f]_\beta^{1-\theta}\right).
\end{aligned}$$

Third case: $\gamma > 1$. We estimate by using the induction hypothesis and the first case

$$\begin{aligned}
[f]_\gamma &\le C\left([f]_1 + [f]_1^{\frac{\beta-\gamma}{\beta-1}} [f]_\beta^{\frac{\gamma-1}{\beta-1}}\right) \\
&\le C\Big(C\left([f]_\alpha + [f]_\alpha^{\frac{\beta-1}{\beta-\alpha}} [f]_\beta^{\frac{1-\alpha}{\beta-\alpha}}\right) + \\
&\quad + C^{\frac{\beta-\gamma}{\beta-1}} \left([f]_\alpha + [f]_\alpha^{\frac{\beta-1}{\beta-\alpha}} [f]_\beta^{\frac{1-\alpha}{\beta-\alpha}}\right)^{\frac{\beta-\gamma}{\beta-1}} [f]_\beta^{\frac{\gamma-1}{\beta-1}}\Big) \\
&\le C^2\,[f]_\alpha + C^2\,[f]_\alpha^{\frac{\beta-1}{\beta-\alpha}} [f]_\beta^{\frac{1-\alpha}{\beta-\alpha}} + \\
&\quad + C^{1+\frac{\beta-\gamma}{\beta-1}} [f]_\alpha^{\frac{\beta-\gamma}{\beta-1}} [f]_\beta^{\frac{\gamma-1}{\beta-1}} + C^{1+\frac{\beta-\gamma}{\beta-1}} [f]_\alpha^\theta [f]_\beta^{1-\theta}.
\end{aligned}$$

The assertion follows from this inequality and two applications of Young's inequality (the same as at the end of the proof of Lemma 1.2 applied to the third term, and a similar one applied to the second term). $\square$

Corollary 1.3. *For every interval $I \subseteq \mathbb{R}$ and $\alpha \in (0,\infty)$ the quantity $\|f\|_\infty + [f]_\alpha$ defines an equivalent norm on $C_b^{\alpha-}(I)$.*

Exercise 2

Let $\mathcal{A} = qD_{xx} + bD_x + c$, where q, b and c are continuous functions over $\mathbb{R}$ with q being positive everywhere.

We want to show that for given $M > 0$ there exist positive constants C, K such that the inequalities

$$\|u'\|_\infty \leq C(\|u\|_\infty + \|\mathcal{A}u\|_\infty)$$

and

$$\|u\|_{C^2([-M,M])} \leq K(\|u\|_\infty + \|\mathcal{A}u\|_\infty)$$

hold for all $u \in C^2([-M,M])$.

In fact, these are consequences of the following statement.

Proposition 2.1. *Let $M > 0$ and define $q_M := \min\{q(x);\, x \in [-M,M]\} > 0$, $b_M := \max\{|b(x)|;\, x \in [-M,M]\}$ and $c_M := \max\{|c(x)|;\, x \in [-M,M]\}$. Then all $u \in C^2([-M,M])$ satisfy*

$$\|u'\|_\infty \leq C_1\Big(\Big(\frac{1}{M} + \frac{b_M}{q_M} + \sqrt{\frac{c_M}{q_M}}\Big)\|u\|_\infty + \frac{1}{\sqrt{q_M}}\|u\|_\infty^{\frac{1}{2}}\|\mathcal{A}u\|_\infty^{\frac{1}{2}}\Big), \tag{2.1}$$

where the constant $C_1 > 0$ does not depend on q, b, c, M or u; and

$$\|u''\|_\infty \leq K\Big(\frac{1}{M}, \frac{b_M}{q_M}, \sqrt{\frac{c_M}{q_M}}\Big)\|u\|_\infty + \frac{2}{q_M}\|\mathcal{A}u\|_\infty, \tag{2.2}$$

where $K\colon [0,\infty)^3 \to [0,\infty)$ is a second order polynomial given explicitly by $K(x,y,z) = C_2(xy + (y+z)^2)$ for a constant $C_2 > 0$ not depending on q, b, c, M or u.

Proof. Let us first prove (2.1). From Exercise 1 we know that there exists a constant C_0, not depending on M or u such that

$$\|u'\|_\infty \leq C_0\Big(\frac{1}{M}\|u\|_\infty + \|u\|_\infty^{\frac{1}{2}}\|u''\|_\infty^{\frac{1}{2}}\Big)$$

is true for all $u \in C^2([-M,M])$. The right-hand side increases if one replaces u'' by qu''/q_M. Therefore we have

$$\begin{aligned}\|u'\|_\infty &\leq C_0\Big(\frac{1}{M}\|u\|_\infty + \frac{1}{\sqrt{q_M}}\|u\|_\infty^{\frac{1}{2}}(\|\mathcal{A}u\|_\infty + b_M\|u'\|_\infty + c_M\|u\|_\infty)^{\frac{1}{2}}\Big)\\ &\leq C_0\Big(\frac{1}{M}\|u\|_\infty + \frac{1}{\sqrt{q_M}}\|u\|_\infty^{\frac{1}{2}}\big(\|\mathcal{A}u\|_\infty^{\frac{1}{2}} + b_M^{\frac{1}{2}}\|u'\|_\infty^{\frac{1}{2}} + c_M^{\frac{1}{2}}\|u\|_\infty^{\frac{1}{2}}\big)\Big).\end{aligned} \tag{2.3}$$

Next we use the inequality

$$C_0\sqrt{\frac{b_M}{q_M}}\|u\|_\infty^{\frac12}\|u'\|_\infty^{\frac12} = \frac12\cdot 2\cdot\Big(C_0^2\frac{b_M}{q_M}\|u\|_\infty\Big)^{\frac12}\|u'\|_\infty^{\frac12}$$
$$\le \frac12\Big(\frac{C_0^2 b_M}{q_M}\|u\|_\infty + \|u'\|_\infty\Big)$$

to absorb the first order derivative of u, appearing in (2.3), into the left-hand side of the inequality. This yields the desired estimate:

$$\|u'\|_\infty \le 2C_0\Big(\Big(\frac{1}{M}+\frac{C_0 b_M}{2q_M}+\sqrt{\frac{c_M}{q_M}}\Big)\|u\|_\infty + \frac{1}{\sqrt{q_M}}\|u\|_\infty^{\frac12}\|\mathcal{A}u\|_\infty^{\frac12}\Big).$$

Let us now prove (2.2). Clearly

$$\|u''\|_\infty \le \frac{1}{q_M}\big(\|\mathcal{A}u\|_\infty + b_M\|u'\|_\infty + c_M\|u\|_\infty\big).$$

Now we use (2.1) to estimate the norm of the first order derivative. If we use in addition the inequality

$$\frac{C_1 b_M}{q_M\sqrt{q_M}}\|u\|_\infty^{\frac12}\|\mathcal{A}u\|_\infty^{\frac12} \le \frac{C_1^2 b_M^2}{4q_M^2}\|u\|_\infty + \frac{1}{q_M}\|\mathcal{A}u\|_\infty$$

it is not difficult to verify that we indeed derive (2.2) with a polynomial K given by $K(x,y,z) = C_1xy + C_1(1+\frac{C_1}{4})y^2 + C_1yz + z^2$. □

The first desired inequality follows from (2.1) via an application of Young's inequality. Having now estimates for all derivatives of $u \in C^2([-M,M])$, we can just sum them up to get the second claimed inequality.

Exercise 3

Let $q, c\colon \mathbb{R}\to(0,\infty)$ and $b\colon \mathbb{R}\to\mathbb{R}$ be continuous.

Proposition 3.1. *Let $u \in C^2(\mathbb{R})$ satisfy $qu''+bu'-cu=0$. Then every local maximum of u is non-positive.*

Proof. Let u attain a local maximum at $x_0 \in \mathbb{R}$. Then we have $u'(x_0)=0$ and $u''(x_0)\le 0$, which imply

$$c(x_0)u(x_0) = q(x_0)u''(x_0) + b(x_0)u'(x_0) = q(x_0)u''(x_0) \le 0,$$

hence, $u(x_0)\le 0$. □

Corollary 3.2. *Let $u \in C^2(\mathbb{R})$ and $x_0, x_1 \in \mathbb{R}$ with $x_0 \ne x_1$ such that the equation $qu''+bu'-cu=0$ holds and $u(x_0)=u(x_1)=0$. Then $u=0$.*

Proof. First, the continuous function u must attain a maximum and a minimum between x_0 and x_1. By the previous result, the maximum is non-positive and the minimum non-negative, hence, u has to be 0 between x_0 and x_1. In particular, we get $u(x_0) = u'(x_0) = 0$. Since the linear ODE

$$v'' = \frac{c}{q}v - \frac{b}{q}v', \quad v(x_0) = v'(x_0) = 0$$

has a unique solution, we conclude $u = 0$. $\square$

Exercise 4

Let $\mathcal{A} = qD_{xx} + bD_x$, where q and b are continuous functions over $\mathbb{R}$ with q being positive everywhere, and $\lambda \in \mathbb{R}$. As in the lecture we define

$$\mathcal{W}(x) := \exp\Big(-\int_0^x \frac{b(s)}{q(s)}\,\mathrm{d}s\Big) \qquad (x \in \mathbb{R}).$$

It is easy to see that for all $x \in \mathbb{R}$ we have $\mathcal{W}(x) \neq 0$ and $\mathcal{W}'(x) = -\mathcal{W}(x)\frac{b(x)}{q(x)}$. Taking now $u \in C^2(\mathbb{R})$, we use this observation to obtain

$$\Big(\frac{u'(x)}{\mathcal{W}(x)}\Big)' = \frac{u''(x)\mathcal{W}(x) - u'(x)\mathcal{W}'(x)}{\mathcal{W}(x)^2} = \frac{q(x)u''(x) + b(x)u'(x)}{q(x)\mathcal{W}(x)} \qquad (x \in \mathbb{R}).$$

From this equality one immediately sees that u is a solution of $\lambda u - \mathcal{A}u = 0$ if and only if the equation

$$\Big(\frac{u'(x)}{\mathcal{W}(x)}\Big)' = \frac{\lambda u(x)}{q(x)\mathcal{W}(x)} \qquad (x \in \mathbb{R}) \tag{4.1}$$

holds. This proves the first part of the exercise.

We now want to show the second part. Assume u solves the problem $\lambda u - \mathcal{A}u = 0$. Integrating both sides of (4.1) yields

$$\int_0^x \left(\frac{u'(s)}{\mathcal{W}(s)}\right)' \mathrm{d}s = \lambda \int_0^x \frac{u(s)}{q(s)\mathcal{W}(s)}\,\mathrm{d}s.$$

We can evaluate the integral on the left side and get

$$\frac{u'(x)}{\mathcal{W}(x)} - \frac{u'(0)}{\mathcal{W}(0)} = \lambda \int_0^x \frac{u(s)}{q(s)\mathcal{W}(s)}\,\mathrm{d}s.$$

By definition $\mathcal{W}(0) = 1$ and therefore

$$u'(x) = \mathcal{W}(x)\left(u'(0) + \lambda \int_0^x \frac{u(s)}{q(s)\mathcal{W}(s)}\,\mathrm{d}s\right).$$

Exercise 5

Prove that $\bar{u}_1\bar{u}_2' - \bar{u}_1'\bar{u}_2 = w_0\mathcal{W}$ on $\mathbb{R}$ for some positive constant w_0.

First, we recall that, as in Lemma 14.2.4, $\bar{u}_1$ denotes a positive, decreasing solution and $\bar{u}_2$ a positive, increasing solution of the equation

$$-q(x)u''(x) - b(x)u'(x) + \lambda u(x) = 0, \quad x \in \mathbb{R},$$

where $q\colon \mathbb{R} \to (0,\infty)$, $b\colon \mathbb{R} \to \mathbb{R}$ are continuous and $\lambda > 0$. Furthermore $\mathcal{W}\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{W}(x) = \exp\Big(-\int_0^x \frac{b(s)}{q(s)}\,\mathrm{d}s\Big) \qquad (x \in \mathbb{R}).$$

We will show the following more general result.

Proposition 5.1. *Let $q\colon \mathbb{R} \to (0,\infty)$ and $b, c\colon \mathbb{R} \to \mathbb{R}$ be continuous, and u_1, u_2 be solutions of the equation $qu'' + bu' + cu = 0$. Then there exists $w_0 \in \mathbb{R}$ such that $u_1u_2' - u_1'u_2 = w_0\mathcal{W}$, where $\mathcal{W}$ is defined as above.*

Proof. Let $w := u_1u_2' - u_1'u_2$. Then we obtain

$$\begin{aligned} w' &= u_1'u_2' + u_1u_2'' - u_1''u_2 - u_1'u_2' = u_1u_2'' - u_1''u_2 \\ &= u_1\frac{-bu_2' - cu_2}{q} - \frac{-bu_1' - cu_1}{q}u_2 \\ &= -\frac{b}{q}(u_1u_2' - u_1'u_2) = -\frac{b}{q}w, \end{aligned}$$

i.e., the function w is a solution of the linear ODE

$$v' = -\frac{b}{q}v.$$

Consequently, we conclude $w(x) = w(0)\mathcal{W}(x)$ for all $x \in \mathbb{R}$. □

As $\bar{u}_1, \bar{u}_2, \bar{u}_2'$ are positive and $\bar{u}_1'$ is negative, we obtain that

$$w_0 := \bar{u}_1(0)\bar{u}_2'(0) - \bar{u}_1'(0)\bar{u}_2(0) > 0.$$

This proves the original claim.

Exercise 6

Deduce Proposition 14.2.6 from Proposition 14.2.5.

As in the formulation of Proposition 14.2.6 there are several inconsistencies with Proposition 14.2.5 we first present a formulation of (a rectified version of) Proposition 14.2.6:

Proposition 14.2.6. *Let $\lambda > 0$. Then the following properties are satisfied:*

(i) *all the solutions u to the equation $\lambda u - \mathcal{A}u = 0$ admit a finite limit at $-\infty$ if and only if $\mathcal{R} \in L^1(-\infty, 0)$;*

(ii) *if $\mathcal{P} \in L^1(-\infty, 0)$ and $\mathcal{R} \notin L^1(-\infty, 0)$, then every positive increasing solution u to the equation $\lambda u - \mathcal{A}u = 0$ satisfies $\lim_{x\to-\infty} \frac{u'(x)}{\mathcal{W}(x)} = 0$;*

(iii) *if $\mathcal{P}, \mathcal{R} \in L^1(-\infty, 0)$, then every solution u to the equation $\lambda u - \mathcal{A}u = 0$ is such that u and $u'/\mathcal{W}$ admit finite limits at $-\infty$. Moreover, there exist two increasing solutions u_1 and u_2 of $\lambda u - \mathcal{A}u = 0$ such that*

$$\lim_{x\to-\infty} u_j(x) = j - 1, \qquad \lim_{x\to-\infty} \frac{u_j'(x)}{\mathcal{W}(x)} = 2 - j \qquad (j = 1, 2);$$

(iv) *there exists a positive increasing solution u to the equation $\lambda u - \mathcal{A}u = 0$ with $\lim_{x\to-\infty} u(x) > 0$, if and only if $\mathcal{P} \in L^1(-\infty, 0)$.*

We recall that $\mathcal{A}$ is the differential operator defined by $\mathcal{A}u = qu'' + bu'$, with $q, b \in C^\alpha_{\text{loc}}(\mathbb{R})$ and $q(x) > 0$ for all $x \in \mathbb{R}$. We define $\hat{q}, \hat{b}$ by

$$\hat{q}(x) := q(-x), \qquad \hat{b}(x) := -b(-x) \qquad (x \in \mathbb{R}),$$

and we denote by $\hat{\mathcal{A}}$ the corresponding differential operator. If $u \in C^2(\mathbb{R})$, and we define

$$\hat{u}(x) := u(-x) \qquad (x \in \mathbb{R}),$$

then

$$\hat{u}'(x) = -u'(-x), \qquad \hat{u}''(x) = u''(-x) \qquad (x \in \mathbb{R}),$$

and therefore

$$\lambda u - \mathcal{A}u = 0 \qquad \text{if and only if} \qquad \lambda\hat{u} - \hat{\mathcal{A}}\hat{u} = 0.$$

Also, u is increasing if and only if $\hat{u}$ is decreasing.

If we define the functions $\hat{\mathcal{W}}, \hat{\mathcal{P}}, \hat{\mathcal{R}}$ corresponding to the operator $\hat{\mathcal{A}}$, then an easy computation shows that

$$\hat{\mathcal{W}}(x) = \mathcal{W}(-x), \qquad \hat{\mathcal{P}}(x) = -\mathcal{P}(-x), \qquad \hat{\mathcal{R}}(x) = -\mathcal{R}(-x) \qquad (x \in \mathbb{R}).$$

This implies that

$$\mathcal{W} \in L^1(-\infty, 0) \qquad \text{if and only if} \qquad \hat{\mathcal{W}} \in L^1(0, \infty),$$

and the corresponding property holds for $\mathcal{P}$ and $\mathcal{R}$.

In view of these observations the properties of u on $(-\infty, 0)$, stated in Proposition 14.2.6, immediately translate to the corresponding properties of $\hat{u}$ on $(0, \infty)$, stated and proved in Proposition 14.2.5.

Exercise 7

Prove that for the operator $\mathcal{A}$ given by

$$(\mathcal{A}\psi)(x) := \psi''(x) + x^3\psi'(x) \qquad (x \in \mathbb{R})$$

on smooth functions ψ, the points $+\infty$ and $-\infty$ are both exit points.

Using the notation of Section 14.2.1. we have

$$q(x) = 1, \qquad b(x) = x^3, \qquad c(x) = 0 \qquad (x \in \mathbb{R}).$$

Hence, the functions $\mathcal{W}, \mathcal{R}$ and $\mathcal{P}$ are given by

$$\begin{aligned}
\mathcal{W}(x) &= \exp\Big(-\int_0^x s^3 \,\mathrm{d}s\Big) = \exp\Big(-\frac{1}{4}x^4\Big) \\
\mathcal{R}(x) &= \mathcal{W}(x)\int_0^x \frac{1}{\mathcal{W}(s)}\,\mathrm{d}s = \int_0^x \exp\Big(\frac{1}{4}(s^4 - x^4)\Big)\,\mathrm{d}s \\
\mathcal{P}(x) &= \frac{1}{\mathcal{W}(x)}\int_0^x \mathcal{W}(s)\,\mathrm{d}s = \int_0^x \exp\Big(\frac{1}{4}(x^4 - s^4)\Big)\,\mathrm{d}s
\end{aligned}$$

for $x \in \mathbb{R}$. We have to show that $\mathcal{R} \in L_1(0,\infty) \cap L_1(-\infty,0)$ and $\mathcal{P} \notin L_1(0,\infty) \cup L_1(-\infty,0)$. Using that

$$\mathcal{R}(-x) = \mathcal{R}(x), \qquad \mathcal{P}(-x) = \mathcal{P}(x) \qquad (x \in \mathbb{R}),$$

it suffices to prove that $\mathcal{R} \in L_1(0,\infty)$ and $\mathcal{P} \notin L_1(0,\infty)$. Since

$$\mathcal{P}(x) = \int_0^x \exp\Big(\frac{1}{4}(x^4 - s^4)\Big)\,\mathrm{d}s \geq \int_0^x \mathrm{d}s = x \qquad (x \geq 0),$$

we infer that $\mathcal{P} \notin L_1(0,\infty)$. Moreover, for $x \geq 1$ we can estimate

$$\begin{aligned}
\mathcal{R}(x) &= \int_0^x \exp\Big(\frac{1}{4}(s^4 - x^4)\Big)\,\mathrm{d}s \\
&= \int_0^{x-x^{-2}} \exp\Big(\frac{1}{4}(s^4 - x^4)\Big)\,\mathrm{d}s + \int_{x-x^{-2}}^x \exp\Big(\frac{1}{4}(s^4 - x^4)\Big)\,\mathrm{d}s \\
&\leq x\exp\Big(\frac{1}{4}\Big(\Big(x - \frac{1}{x^2}\Big)^4 - x^4\Big)\Big) + x - \Big(x - \frac{1}{x^2}\Big) \\
&= x\exp\Big(-x + \frac{3}{2x^2} - \frac{1}{x^5} + \frac{1}{4x^8}\Big) + \frac{1}{x^2} \\
&\leq x\exp\Big(-x + \frac{7}{4}\Big) + \frac{1}{x^2}
\end{aligned}$$

or alternatively

$$\begin{aligned}\mathcal{R}(x) &= \exp\Big(-\frac{1}{4}x^4\Big)\int_0^{x/2}\exp\Big(\frac{1}{4}s^4\Big)\,\mathrm{d}s + \exp\Big(-\frac{1}{4}x^4\Big)\int_{x/2}^{x}\exp\Big(\frac{1}{4}s^4\Big)\,\mathrm{d}s \\ &\leq \frac{x}{2}\exp\Big(-\frac{1}{4}x^4 + \frac{1}{64}x^4\Big) + \frac{8}{x^3}\exp\Big(-\frac{1}{4}x^4\Big)\int_{x/2}^{x} s^3\exp\Big(\frac{1}{4}s^4\Big)\,\mathrm{d}s \\ &\leq \frac{x}{2}\exp\Big(-\frac{15}{64}x^4\Big) + \frac{8}{x^3}.\end{aligned}$$

Thus, we obtain that $\mathcal{R} \in L_1(1,\infty)$. Moreover, using that $\mathcal{R}$ is continuous, we derive $\mathcal{R} \in L_1(0,\infty)$.