ISEM 20 — Exercises in Lecture 12

Solutions by the Wuppertal-Team

Exercise 1

Define the operator $\mathcal{A}_0$ by $\mathcal{A}_0 w = \mathcal{A}w - cw$. For $c_0 := \|c\|_\infty$ consider the function v given by $v(t,x) = e^{-(c_0+1)t}u(t,x) - \|g\|_\infty$. Then

$$\begin{aligned}
D_t v(t,x) &= -(c_0+1)e^{-(c_0+1)t}u(t,x) + e^{-(c_0+1)t}D_t u(t,x)\\
&= -(c_0+1)v(t,x) - (c_0+1)\|g\|_\infty + e^{-(c_0+1)t}(\mathcal{A}u(t,x) + g(t,x))\\
&= -(c_0+1)v(t,x) - (c_0+1)\|g\|_\infty + \mathcal{A}_0 v(t,x) + ce^{-(c_0+1)t}u(t,x) - c\|g\|_\infty + e^{-(c_0+1)t}g(t,x)\\
&= -(c_0+1-c)v(t,x) + \mathcal{A}_0 v(t,x) - (c_0+1-c)\|g\|_\infty + e^{-(c_0+1)t}g(t,x)\\
&\le -(c_0+1-c)v(t,x) + \mathcal{A}_0 v(t,x) - (c_0+1-c)\|g\|_\infty + \|g\|_\infty\\
&\le -(c_0+1-c)v(t,x) + \mathcal{A}_0 v(t,x).
\end{aligned}$$

Since $-(c_0+1-c) \le 0$, by the weak maximum principle (potential term ≤ 0, Theorem 1.1.2 (i)) we obtain

$$v(t,x) \le \sup_{(s,y)\in\Gamma_T} v^+(s,y)$$

which implies

$$u(t,x) \le e^{(c_0+1)t} \sup_{(s,y)\in\Gamma_T} v^+(s,y) + e^{(c_0+1)t}\|g\|_\infty \le C_0\Big(\sup_{(s,y)\in\Gamma_T} |v|(s,y) + \|g\|_\infty\Big) \le C_0\Big(\sup_{(s,y)\in\Gamma_T} |u|(s,y) + 2\|g\|_\infty\Big)$$

for each $t, x \in \overline{\Omega_T}$ and for the number $C_0 = e^{(c_0+1)T}$. From this we conclude

$$u(t,x) \le C_0(\|f\|_\infty + 2\|g\|_\infty). \tag{1}$$

If u is a solution of the equation under consideration, then $-u$ solves the "same" equation with the different initial and boundary values: $-f$ and $-g$. So that

$$-u(t,x) \le C_0(\|-f\|_\infty + 2\|-g\|_\infty),$$

and this, together with (1), completes the proof of the desired inequality in (12.15) with $C = 2C_0$. ∎

Exercise 2

By assumption $f(x,y) = u(x)v(y)$ for $(x,y) \in \overline{\Omega}_1 \times \overline{\Omega}_2$ and for some $u \in C_b(\overline{\Omega}_1)$, $v \in C_b^\theta(\overline{\Omega}_2)$, where $\Omega_1 \subseteq \mathbb{R}^M, \Omega_2 \subseteq \mathbb{R}^N$ open and $\theta \in (0,1)$.

First, we show that $f(x,\cdot) \in C_b^\theta(\overline{\Omega}_2)$ for all $x \in \overline{\Omega}_1$, i.e. $\|f(x,\cdot)\|_{C_b^\theta(\overline{\Omega}_2)} < \infty$ for all $x \in \overline{\Omega}_1$. Note that

$$C_b^\theta(\overline{\Omega}_2) = \left\{ f : \overline{\Omega}_2 \to \mathbb{R} :\ f \text{ is bounded and } \theta\text{-Hölder-continuous} \right\}.$$

Let $x \in \overline{\Omega}_1$ be fixed. Thus, by assumption we obtain

$$\|f(x,\cdot)\|_{C_b^\theta(\overline{\Omega}_2)} = \|u(x)v(\cdot)\|_{C_b^\theta(\overline{\Omega}_2)} = |u(x)|\, \|v\|_{C_b^\theta(\overline{\Omega}_2)} < \infty.$$

Moreover, we have

$$\begin{aligned}
\sup_{x\in\overline{\Omega}_1} [f(x,\cdot)]_{C_b^\theta(\overline{\Omega}_2)} &= \sup_{x\in\overline{\Omega}_1} \sup_{\substack{y,z\in\overline{\Omega}_2\\ y\neq z}} \frac{|f(x,y)-f(x,z)|}{|y-z|^\theta}\\
&= \sup_{x\in\overline{\Omega}_1} \sup_{\substack{y,z\in\overline{\Omega}_2\\ y\neq z}} |u(x)| \frac{|v(y)-v(z)|}{|y-z|^\theta}\\
&= \sup_{x\in\overline{\Omega}_1} |u(x)| \cdot \sup_{\substack{y,z\in\overline{\Omega}_2\\ y\neq z}} \frac{|v(y)-v(z)|}{|y-z|^\theta}\\
&= \|u\|_\infty \cdot [v]_{C_b^\theta(\overline{\Omega}_2)}\,.
\end{aligned}$$

∎

Exercise 3

To be proven is:

Lemma 12.2.1

(ii) Let $T > 0$ be fixed. There is a positive constant C such that for every $u \in C_b^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\mathbb{R}^d_+})$ with $u(0,\cdot) = 0$ on $\overline{\mathbb{R}^d_+}$, $u(t,\cdot) = 0$ on $\partial\mathbb{R}^d_+$ for all $t \in [0,T]$ the inequality

$$\lambda^{1-\frac{\alpha}{2}} \|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} + \lambda^{\frac{1-\alpha}{2}} \sum_{i=1}^d \|D_i u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} \le C \|D_t u - \mathcal{A}u + \lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} \tag{2}$$

holds for every $\lambda \ge 1$.

Proof. As recommended in the lecture, we follow the lines of the proof of the first part of Lemma 12.2.1, but we add some details and comments that arose during our discussions. We define $v : [0,T] \times \overline{\mathbb{R}^{d+1}_+} \to \mathbb{R}$ by

$$v(t,y,x) = \cos(\sqrt{\lambda}y)u(t,x)$$

for $t \in [0,T]$, $y \in \mathbb{R}$ and $x \in \overline{\mathbb{R}^d_+}$, where $\overline{\mathbb{R}^{d+1}_+} := \mathbb{R} \times \mathbb{R}^{d-1} \times [0,\infty)$. As before, it is not difficult to show that v belongs to $C_b^{1+\alpha/2,2+\alpha}([0,T] \times \overline{\mathbb{R}^{d+1}_+})$. Let us define the operator $\mathcal{A}'$ on sufficiently smooth functions $\zeta : \overline{\mathbb{R}^{d+1}_+} \to \mathbb{R}$ (now on the half-space $\overline{\mathbb{R}^{d+1}_+}$) by

$$\mathcal{A}'\zeta(y,x) := D_{yy}\zeta(y,x) + \sum_{i,j=1}^d q_{ij}(x)D_{ij}\zeta(y,x) + \sum_{j=1}^d b_j(x)D_j\zeta(y,x) + c(x)\zeta(y,x) \quad (y,x) \in \overline{\mathbb{R}^{d+1}_+}.$$

This operator is again elliptic with ellipticity constant 1 or μ, depending on which is smaller (μ is the ellipticity constant of $\mathcal{A}$).

The crucial point in the proof will be the a priori estimate

$$\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})} \le C \|D_t v - \mathcal{A}'v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})}. \tag{3}$$

This is not covered in the lectures, but some analogous statements are: Theorems 7.2.1 and 11.2.4. After a careful analysis of the proof of Theorem 7.2.1 we do not see any problem to modify that proof for this situation. Furthermore, in Theorem 11.2.4 we obtain another a priori estimate, again not perfectly suited for our situation, since the underlying space is $(-\infty,T] \times \overline{\mathbb{R}^{d+1}_+}$ and not $[0,T] \times \overline{\mathbb{R}^{d+1}_+}$. It appears that this statement could be used for the proof of (3), but this proof using cut-off functions would not be very different from the one that goes along the lines of Theorem 7.2.1. Anyway, we think we have enough evidence for the validity of (3), hence we leave its proof to the reader, and focus only on the explanation of the proof of Lemma 12.2.1(ii).
Observing that

$$D_t v(t,y,x) - \mathcal{A}'v(t,y,x) = \big(D_t u(t,x) - \mathcal{A}u(t,x) + \lambda u(t,x)\big)\cos(\sqrt{\lambda}y),$$

then Exercise 12.2 implies

$$\begin{aligned}
\|D_t v - \mathcal{A}'v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})} &\le \|D_t u - \mathcal{A}u + \lambda u\|_\infty + \|\cos(\sqrt{\lambda}\cdot)\|_\infty \sup_{x\in\overline{\mathbb{R}^d_+}} [D_t u(\cdot,x) - \mathcal{A}u(\cdot,x) + \lambda u(\cdot,x)]_{C^{\alpha/2}([0,T])} \\
&\quad + \|\cos(\sqrt{\lambda}\cdot)\|_\infty \sup_{t\in[0,T]} [D_t u(t,\cdot) - \mathcal{A}u(t,\cdot) + \lambda u(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})} \\
&\quad + [\cos(\sqrt{\lambda}\cdot)]_{C_b^\alpha(\mathbb{R})} \|D_t u - \mathcal{A}u + \lambda u\|_\infty \\
&\le \|D_t u - \mathcal{A}u + \lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} + C\lambda^{\frac{\alpha}{2}} \|D_t u - \mathcal{A}u + \lambda u\|_\infty
\end{aligned} \tag{4}$$

for some constant $C \ge 1$ (details are in Explanation 1 below). Hence we obtain

$$\|D_t v - \mathcal{A}'v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})} \le 2C\lambda^{\frac{\alpha}{2}} \|D_t u - \mathcal{A} + \lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}, \tag{5}$$

since the $\|\cdot\|_\infty$-norm also occurs in the first term in (4) and since $\lambda \ge 1$.

We correct a probable typo in the lecture notes in the third line of the displayed formula after (12.19): we take the $\|\cdot\|_{C^{\alpha/2}([0,T])}$-norm instead of the $[\cdot]_{C^{\alpha/2}([0,T])}$-seminorm. In our situation of the half-space the following holds:

$$\lambda \|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} + C_0\lambda^{\frac{\alpha}{2}} \sum_{i,j=1}^d \|D_{ij}u\|_\infty + \sqrt{\lambda}\sum_{i=1}^d \sup_{x\in\overline{\mathbb{R}^d_+}} \|D_i u(\cdot,x)\|_{C^{\alpha/2}([0,T])}$$

$$\leq \|D_{yy}v(\cdot,0,\cdot)\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} + \sum_{i,j=1}^d \sup_{(t,x)\in[0,T]\times\overline{\mathbb{R}^d_+}} [D_{ij}v(t,\cdot,x)]_{C_b^\alpha(\mathbb{R})} + \sum_{i=1}^d \sup_{(y,x)\in\overline{\mathbb{R}^{d+1}_+}} \|D_{yi}v(\cdot,y,x)\|_{C^{\alpha/2}([0,T])},$$

$$\leq \|v\|_\infty + \sum_{i,j=1}^d \|D_{ij}v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})} + \sum_{i=1}^d \|D_{yi}v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})} + \|D_{yy}v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})}$$

$$+ \sum_{i=1}^d \|D_jv\|_\infty + \|D_yv\|_\infty + \|D_tv\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})}, \tag{6}$$

$$= \|v\|_\infty + \sum_{i,j=1}^{d+1} \|D_{ij}v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})} + \sum_{i=1}^{d+1} \|D_jv\|_\infty + \|D_tv\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})},$$

$$= \|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})}. \tag{7}$$

The first estimate follows from calculating the derivative in the y-coordinate and pulling the λ, $|\sin(\sqrt{\lambda}\cdot)|$ or $|\cos(\sqrt{\lambda}\cdot)|$ out of the norm/seminorm. In the last inequality (6) we compare single terms in the expressions as described in detail below in Explanation 2.

By inserting inequality (5) and (7) in (3) we conclude that

$$\lambda^{1-\frac{\alpha}{2}} \|u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} + \lambda^{\frac{1-\alpha}{2}} \sum_{i=1}^d \sup_{x\in\overline{\mathbb{R}^d_+}} \|D_iu(\cdot,x)\|_{C^{\alpha/2}([0,T])} + C_0 \sum_{i,j=1}^d \|D_{ij}u\|_\infty \tag{8}$$
$$\leq 2C\|D_tu - \mathcal{A} + \lambda u\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}.$$

To get the desired inequality (2), we need to estimate $\|D_iu\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})}$, in particular the term $[D_iu(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})}$:

$$\|D_iu\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} = \|D_iu\|_{C_b^{0,\alpha}([0,T]\times\overline{\mathbb{R}^d_+})} + \sup_{x\in\overline{\mathbb{R}^d_+}} [D_iu(\cdot,x)]_{C^{\alpha/2}([0,T])},$$
$$= \sup_{t\in[0,T]} \left(\|D_iu\|_\infty + [D_iu(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})}\right) + \sup_{x\in\overline{\mathbb{R}^d_+}} [D_iu(\cdot,x)]_{C^{\alpha/2}([0,T])}.$$

At this point we need the following interpolation estimate:[1]

$$\|\zeta\|_{C_b^{1+\alpha}(\overline{\mathbb{R}^d_+})} \leq C_\alpha \|\zeta\|_{C_b^\alpha(\overline{\mathbb{R}^d_+})}^{\frac{1-\alpha}{2-\alpha}} \|\zeta\|_{C_b^2(\overline{\mathbb{R}^d_+})}^{\frac{1}{2-\alpha}}, \tag{9}$$

which holds, for positive constant C_α depending only on α. We expand the inequality to see where it can be used:

$$\sum_{i=1}^d [D_iu(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})} \leq \|u(t,\cdot)\|_\infty + \sum_{i=1}^d \|D_iu(t,\cdot)\|_\infty + \sum_{i=1}^d [D_iu(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})} = \|u(t,\cdot)\|_{C_b^{1+\alpha}(\overline{\mathbb{R}^d_+})}$$
$$\leq C_\alpha\Big(\|u(t,\cdot)\|_\infty + [u(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})}\Big)^{\frac{1-\alpha}{2-\alpha}} \Big(\|u(t,\cdot)\|_\infty + \sum_{i=1}^d \|D_iu(t,\cdot)\|_\infty + \sum_{i,j=1}^d \|D_{ij}u(t,\cdot)\|_\infty\Big)^{\frac{1}{2-\alpha}}. \tag{10}$$

We multiply by λ^ε and $(1/\lambda)^\varepsilon$ where $\varepsilon = (1-\alpha)/(2(2-\alpha))$ and then apply Young's inequality with $p = (2-\alpha)/(1-\alpha)$ and $q = 2-\alpha$ to obtain:

$$\sum_{i=1}^d [D_iu(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})}$$
$$\leq C_\alpha\lambda^\varepsilon\Big(\|u(t,\cdot)\|_\infty + [u(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})}\Big)^{\frac{1-\alpha}{2-\alpha}} \frac{1}{\lambda^\varepsilon}\Big(\|u(t,\cdot)\|_\infty + \sum_{i=1}^d \|D_iu(t,\cdot)\|_\infty + \sum_{i,j=1}^d \|D_{ij}u(t,\cdot)\|_\infty\Big)^{\frac{1}{2-\alpha}}$$
$$\leq C_\alpha\left(\frac{\lambda^{\varepsilon p}\left(\|u(t,\cdot)\|_\infty + [u(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})}\right)^{\frac{1-\alpha}{2-\alpha}p}}{p} + \frac{\left(\|u(t,\cdot)\|_\infty + \sum_{i=1}^d \|D_iu(t,\cdot)\|_\infty + \sum_{i,j=1}^d \|D_{ij}u(t,\cdot)\|_\infty\right)^{\frac{q}{2-\alpha}}}{\lambda^{q\varepsilon}q}\right)$$
$$\leq C_\alpha\Big(\lambda^{\frac{1}{2}}\Big(\|u(t,\cdot)\|_\infty + [u(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^d_+})}\Big) + \lambda^{\frac{\alpha-1}{2}}\Big(\|u(t,\cdot)\|_\infty + \sum_{i=1}^d \|D_iu(t,\cdot)\|_\infty + \sum_{i,j=1}^d \|D_{ij}u(t,\cdot)\|_\infty\Big)\Big).$$

[1] This is not proved in the lectures.

Taking into account that $\lambda \geq 1$, we conclude, with a different C, from (8) that

$$\begin{aligned}\sum_{i=1}^{d}[D_i u(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}_+^d})} &\leq C\left\|D_t u - \mathcal{A}u + \lambda u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})} + C\lambda^{-\frac{1-\alpha}{2}}\left\|D_t u - \mathcal{A}u + \lambda u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})} \\ &\leq 2C\lambda^{-\frac{1-\alpha}{2}}\left\|D_t u - \mathcal{A}u + \lambda u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})}. \end{aligned} \tag{11}$$

Using again (8) one obtains

$$\sum_{i=1}^{d}\sup_{x\in\overline{\mathbb{R}_+^d}}\left\|D_i u(\cdot,x)\right\|_{C^{\alpha/2}([0,T])} \leq C\lambda^{-\frac{1-\alpha}{2}}\left\|D_t u - \mathcal{A}u + \lambda u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})},$$

and from equation (11) follows

$$\sum_{i=1}^{d}[D_i u(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}_+^d})} \leq 2C\lambda^{-\frac{1-\alpha}{2}}\left\|D_t u - \mathcal{A}u + \lambda u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})}.$$

Note that the constant C differs in the two inequalities. But eventually, with the agreement of constantly changing constants, we obtain:

$$\begin{aligned}\sum_{j=1}^{d}\left\|D_j u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})} &= \sum_{i=1}^{d}\sup_{x\in\overline{\mathbb{R}_+^d}}\left\|D_i u(\cdot,x)\right\|_{C^{\alpha/2}([0,T])} + \sum_{i=1}^{d}\sup_{t\in[0,T]}[D_i u(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}_+^d})}, \\ &\leq C\lambda^{-\frac{1-\alpha}{2}}\left\|D_t u - \mathcal{A}u + \lambda u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})}, \end{aligned} \tag{12}$$

and also the desired inequality (2), since we can estimate every single term on the left-hand side of (2) with the help of (8) and (12) (C is changed once more):

$$\lambda^{1-\frac{\alpha}{2}}\left\|u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})} + \lambda^{\frac{1-\alpha}{2}}\sum_{i=1}^{d}\left\|D_i u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})} \leq C\left\|D_t u - \mathcal{A}u + \lambda u\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})}.$$

Finally, we present some details for some of the estimates; these have turned out to be useful in our discussions.

Explanation 1: The first term in (4) can be estimated easily by pulling out the supremum of $|\cos(\sqrt{\lambda}\cdot)|$. To handle the second term we define

$$V(t,x) := D_t u - \mathcal{A}u + \lambda u, \qquad \rho(x,\tilde{x},y,\tilde{y}) := \left|\left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right) - \left(\begin{smallmatrix} \tilde{y} \\ \tilde{x} \end{smallmatrix}\right)\right|^\alpha, \qquad x,\tilde{x}\in\overline{\mathbb{R}_+^d},\ y,\tilde{y}\in\mathbb{R}.$$

Then we add a zero and estimate:

$$\begin{aligned}&\sup\left\{\frac{\left|V(t,x)\cos(\sqrt{\lambda}y) - V(t,\tilde{x})\cos(\sqrt{\lambda}\tilde{y})\right|}{\rho(x,\tilde{x},y,\tilde{y})} : (y,x),(\tilde{y},\tilde{x})\in\overline{\mathbb{R}_+^{d+1}}\right\} \\ &\leq \sup\left\{\frac{\left|V(t,x) - V(t,\tilde{x})\right|}{\rho(x,\tilde{x},y,\tilde{y})} : (y,x),(\tilde{y},\tilde{x})\in\overline{\mathbb{R}_+^{d+1}}\right\} + \left\|V(t,\cdot)\right\|_\infty \sup\left\{\frac{\left|\cos(\sqrt{\lambda}y) - \cos(\sqrt{\lambda}\tilde{y})\right|}{\rho(x,\tilde{x},y,\tilde{y})} : (x,y),(\tilde{x},\tilde{y})\in\overline{\mathbb{R}_+^{d+1}}\right\}, \\ &= [V(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}_+^d})} + [\cos(\sqrt{\lambda}\cdot)]_{C_b^\alpha(\mathbb{R})}\left\|V\right\|_\infty, = [V(t,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}_+^d})} + \lambda^{\alpha/2}[\cos(\cdot)]_{C_b^\alpha(\mathbb{R})}\left\|V\right\|_\infty. \end{aligned}$$

Taking the $\sup_{t\in[0,T]}$ on both sides we arrive at (4).

Explanation 2: We compare the terms in equation (6) with the terms that occur in $\|v\|_{C_b^{1+\alpha/2,2+\alpha}([0,T]\times\overline{\mathbb{R}_+^{d+1}})}$. It is easy to see that the following estimates hold, and we just need to match the corresponding terms in equation (7):

$$\left\|D_{yy}v(\cdot,0,\cdot)\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^d})} \leq \left\|D_{yy}v(\cdot,\cdot,\cdot)\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^{d+1}})}. \tag{13}$$

$$\begin{aligned}\sum_{i=1}^{d}\sup_{(y,x)\in\overline{\mathbb{R}_+^{d+1}}}\left\|D_{yi}v(\cdot,y,x)\right\|_{C^{\alpha/2}([0,T])} &\leq \sum_{i=1}^{d}\Big(\left\|D_{yi}v\right\|_{C_b^{0,\alpha}([0,T]\times\overline{\mathbb{R}_+^{d+1}})} + \sup_{(y,x)\in\overline{\mathbb{R}_+^{d+1}}}\left\|D_{yi}v(\cdot,y,x)\right\|_{C^{\alpha/2}([0,T])}\Big), \\ &= \sum_{i=1}^{d}\left\|D_{yi}v\right\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}_+^{d+1}})}, \end{aligned} \tag{14}$$

$$
\begin{aligned}
\sum_{i,j=1}^{d} \sup_{(t,x)\in[0,T]\times\overline{\mathbb{R}^d_+}} [D_{ij}v(t,\cdot,x)]_{C_b^\alpha(\mathbb{R})} &\le \sum_{i,j=1}^{d} \Big(\|D_{ij}v\|_{C_b^{0,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})} + \sup_{(y,x)\in\overline{\mathbb{R}^{d+1}_+}} \|D_{ij}v(\cdot,y,x)\|_{C^{\alpha/2}([0,T])} \Big), \\
&= \sup_{t\in[0,T]} \Big(\sum_{i,j=1}^{d} \|D_{ij}v(t,\cdot)\|_\infty + \sum_{i,j=1}^{d} [D_{ij}v(t,\cdot,\cdot)]_{C_b^\alpha(\overline{\mathbb{R}^{d+1}_+})} \Big) \\
&\quad + \sum_{i,j=1}^{d} \sup_{(y,x)\in\overline{\mathbb{R}^{d+1}_+}} \|D_{ij}v(\cdot,y,x)\|_{C^{\alpha/2}([0,T])}, \\
&= \sum_{i,j=1}^{d} \|D_{ij}v\|_{C_b^{\alpha/2,\alpha}([0,T]\times\overline{\mathbb{R}^{d+1}_+})}. \qquad (15)
\end{aligned}
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