Solutions to the Exercises of Lecture 15, Team Hamburg

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Exercise 1

Fix $f \in D(\hat{A})$. Prove the following properties:

(i) $T(t)f \in D(\hat{A})$ for all $t \geq 0$ and $\hat{A}T(t)f = T(t)\hat{A}f$;

(ii) for each $x \in \mathbb{R}^d$, the function $(T(\cdot)f)(x)$ is of class C^1 and $\frac{d}{dt}(T(t)f)(x) = (T(t)\hat{A}f)(x)$;

(iii) $D(\hat{A})$ is dense in $C_b(\mathbb{R}^d)$ with respect to dominated pointwise convergence, i.e. for every $f \in C_b(\mathbb{R}^d)$, there is a sequence of functions $(f_n) \subset D(\hat{A})$ uniformly bounded and converges pointwise to f;

(iv) $(\hat{A}, D(\hat{A}))$ is closed in $C_b(\mathbb{R}^d)$ with respect to the dominated pointwise convergence.

(v) $(c_0, \infty) \subset \rho(\hat{A})$ [1] and for ever $\lambda > c_0,\ f \in C_b(\mathbb{R}^d),\ x \in \mathbb{R}^d$

$$R(\lambda, \hat{A})f(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)dt;$$

(vi) $C_c^\infty(\mathbb{R}^d) \subset D(\hat{A})$ and $\mathcal{A}f = \hat{A}f$ for every $f \in C_c^\infty(\mathbb{R}^d)$.

Let us start by recalling the following facts.

- $T(t)f = u(t, \cdot)$ where u is the unique (bounded) solution to ACP (14.1) (uniqueness is assumed, existence holds by Theorem 14.1.1))
- $T(t+s) = T(t)T(s)$ by uniqueness of the solution
- $\|T(t)\| \leq \mathrm{e}^{c_0 t}$ for $t \geq 0$
- $t \mapsto (T(t)f)(x)$ is continuous on $[0, \infty)$ for any $f \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ (as $u \in C([0, \infty) \times \mathbb{R}^d)$.

Denote a uniformly bounded sequence of functions $(g_n) \subset C_b(\mathbb{R}^d)$ converging pointwise to $g \in C_b(\mathbb{R}^d)$ by $g_n \xrightarrow{\text{bpc}} g$.

Lemma 1 (Continuity of $T(s)$ in *bpc*-topology)**.** *Let $g_n \xrightarrow{bpc} g$ and $s > 0$. Then $T(s)g_n \xrightarrow{bpc} T(s)g$.*

Proof. First we observe that for any $x \in \mathbb{R}^d$, $f \mapsto (T(s)f)(x)$ is a linear, continuous functional on $C_0(\mathbb{R}^d)$ and thus, by the Riesz-Markov representation theorem, takes the form

$$(T(s)f)(x) = \int_{\mathbb{R}^d} f(y)\mu_{s,x}(dy) =: E_{\mu_{s,x}}[f] \qquad \text{for any } f \in C_0(\mathbb{R}^d),$$

where $\mu_{s,x}$ is a finite, complex Radon measure. In fact, we can extend this equality to $f \in C_b(\mathbb{R}^d)$. Recall that in the proof of Theorem 14.1.1, we proved in Step 2 that if (f_n) is uniformly bounded and converges to f locally uniformly, $T(s)f_n \xrightarrow{\text{bpc}} T(s)f$. As C_0 is dense in C_b with respect to uniformly bounded locally uniform convergence, we can choose a sequence $(f_n) \subset C_0(\mathbb{R}^d)$ converging in this topology to f. We have

$$(T(s)f)(x) \leftarrow (T(s)f_n)(x) = E_{\mu_{s,x}}[f_n] \to E_{\mu_{s,x}}[f],$$

as $E_\mu[\cdot]$ is continuous in this topology as well.
Since $g_n \xrightarrow{\text{bpc}} g$ implies that $E_{\mu_{x,s}}[g_n] \to E_{\mu_{s,x}}[g]$ for any x by dominated convergence, and since

$$\sup_n \|T(s)g_n\|_\infty \leq \mathrm{e}^{c_0 t} \sup_n \|g_n\|_\infty < \infty,$$

we obtain that

$$(T(s)g_n) = E_{\mu_{s,\cdot}}[g_n] \xrightarrow{\text{bpc}} E_{\mu_{s,\cdot}}[g] = T(s)g.$$

□

[1] in the exercise, it was asked to prove $(0, \infty) \subset \rho(\hat{A})$, which probably meant the case $c_0 = 0$.

(i): As $f \in D(\hat{A})$, we know that

$$g_t^f := \frac{T(t)f - f}{t} \xrightarrow{\text{bpc}} g = \hat{A}f \quad (t \to 0^+).$$

Let $s \geq 0$. Then $T(s)f \in D(\hat{A})$ and $\hat{A}T(s)f = T(s)\hat{A}f$ is equivalent to

$$g_t^{T(s)f} = T(s)g_t^f \xrightarrow{\text{bpc}} T(s)\hat{A}f.$$

The latter follows directly by Lemma 1.

(ii): Let $x \in \mathbb{R}^d$. $(T(\cdot)f)(x)$ is right-differentiable, as we just proved in (i) that the limit exists and is equal to $(T(\cdot)\hat{A}f)(x)$. On the other hand, $(T(\cdot)\hat{A}f)(x)$ is continuous and as every real-valued right-differentiable function with continuous right-derivative is differentiable, we are done.

(iii): Let $f \in C_b(\mathbb{R}^d)$ be given. Let $n \in \mathbb{N}$. We recall the idea of Prop. 2.1.7 and claim that

$$f_n : x \mapsto n \int_0^{1/n} (T(s)f)(x) ds$$

is in $D(\hat{A})$ and converges to f locally uniformly. First of all, $f_n \in C_b$ as $\|f_n\| \leq \mathrm{e}^{c_0 t}\|f\|$ for all $t \geq 0$. Moreover, we have that f_n is the *bpc*-limit of Riemann sums in C_b. For instance, for $k \in \mathbb{N}$ define

$$F_k := n \sum_{l=0}^{2^k-1} \frac{T\left(\frac{l}{n2^k}\right) f}{n2^k}.$$

Then F_k in $C_b(\mathbb{R}^d)$, $\|F_k\| \leq \mathrm{e}^{c_0 \vee 1}\|f\|$ and $F_k \to f_n$ pointwise as $k \to \infty$. Hence, by Lemma 1, $T(h)f_n = n \int_0^{1/n} (T(h)T(s)f)(\cdot) ds$ and

$$\begin{aligned} g_h^{f_n}(x) := \frac{(T(h)f_n)(x) - f_n(x)}{h} &= \frac{n}{h}\left(T(h)\int_0^{1/n} T(s)f(x)ds - \int_0^{1/n} T(s)f(x)ds\right) \\ &= \frac{n}{h}\left(\int_{1/n}^{1/n+h} T(s)f(x)ds - \int_0^{h} T(s)f(x)ds\right) \quad (x \in \mathbb{R}^d). \end{aligned}$$

The family $(g_h^{f_n})_{h>0}$ is uniformly bounded and converges pointwise to $n(T(1/n)f(x) - f(x))$. We proved that $f_n \in D(\hat{A})$. Now clearly (f_n) is a uniformly bounded sequence, converging pointwise to f (due to the continuity of $(T(\cdot)f)(x)$ for all $x \in \mathbb{R}^d$). Indeed, (f_n) converges even uniformly on compact subsets, which follows by

$$\lim_{t \to 0^+} \sup_{x \in K} |(T(t)f)(x) - f(x)| = 0,$$

for any $f \in C_b(\mathbb{R}^d)$ and $K \subset \mathbb{R}^d$ compact.

(iv): Again, we can mimic the proof of Prop. 2.1.7. Assume that $f_n \in D(\hat{A})$ and $f_n \xrightarrow{\text{bpc}} f$, $\hat{A}f_n \xrightarrow{\text{bpc}} g$. Now, using (i) and (ii), for $x \in \mathbb{R}^d$ and $h > 0$ we have

$$\frac{T(h)f_n(x) - f_n(x)}{h} = \frac{1}{h}\int_0^h T(t)\hat{A}f_n(x)\,dt,$$

which for $n \to \infty$ results in

$$\frac{T(h)f(x) - f(x)}{h} = \frac{1}{h}\int_0^h T(t)g(x)\,dt,$$

and by letting $h \to 0$, we obtain $f \in D(\hat{A})$ and $\hat{A}f(x) = g(x)$.

(v): Now we mimic Prop. 2.1.12. Let $x \in \mathbb{R}^d$. We define for any $\lambda > c_0$

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t}(T(t)f)(x)dt.$$

This is a well-defined, bounded operator as we know that $\|T(t)\| \le e^{(c_0 \vee 1)t}$. We claim that $R_\lambda f(x)$ is the resolvent applied to f at x, i.e. $R(\lambda, A)f(x)$. Indeed, for $f \in D(\hat{A})$ we have

$$\begin{aligned} R_\lambda \hat{A} f(x) &= \lim_{n\to\infty} \int_0^n e^{-\lambda t} \underbrace{(T(t)\hat{A}f)(x)}_{=\frac{d}{dt}(T(t)f)(x)} dt = \lim_{n\to\infty} \left(e^{-\lambda n}(T(n)f)(x) - f(x) + \lambda \int_0^n e^{-\lambda t} T(t)f(x)dt \right) \\ &= \lambda R_\lambda f(x) - f(x). \end{aligned}$$

In particular, $f(x) = R_\lambda(\hat{A} - \lambda I)f(x)$. Now we fix $f \in C_b(\mathbb{R}^d)$. Define

$$F_n := \int_0^n e^{-\lambda t}(T(t)f)(\cdot)dt, \quad F := \int_0^\infty e^{-\lambda t}(T(t)f)(\cdot)dt.$$

Then (F_n) in $C_b(\mathbb{R}^d)$, $F \in C_b(\mathbb{R}^d)$ and $F_n \xrightarrow{\text{bpc}} F$. So we are able to apply Lemma 1 to get

$$T(h)R_\lambda f(x) = \int_0^\infty e^{-\lambda t}(T(t+h)f)(x)dt.$$

Using this we conclude

$$\begin{aligned} \frac{T(h)R_\lambda f(x) - R_\lambda f(x)}{h} &= \frac{1}{h}\left(\int_0^\infty e^{-\lambda t}(T(t+h)f)(x)dt - \int_0^\infty e^{-\lambda t}(T(t)f)(x)dt \right) \\ &= \frac{e^{\lambda h}}{h} \int_h^\infty e^{-\lambda t}(T(t)f)(x)dt - \frac{1}{h}\int_0^h e^{-\lambda t}(T(t)f)(x)dt \\ &= \frac{e^{\lambda h}-1}{h} \int_0^\infty e^{-\lambda t}(T(t)f)(x)dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}(T(t)f)(x)dt \\ &\xrightarrow[h\to 0]{bpc} \lambda R_\lambda f(x) - f(x). \end{aligned}$$

(vi): We assume that $c_0 = 0$ in the following, but the general case holds with straight-forward modifications. Let $f \in C_c^\infty(\mathbb{R}^d)$. By Step 1 of the proof of Theorem 14.1.1 we obtain that $[0,1] \times K \ni (t,x) \mapsto T(t)f(x)$ is in $C^{1,2}([0,1] \times K)$ for any compact $K \subseteq \mathbb{R}^d$. Let $t > 0$, $x \in \mathbb{R}^d$. Then

$$\begin{aligned} \frac{1}{t}\big(T(t)f(x) - f(x)\big) &= \frac{1}{t}\int_0^t \frac{d}{ds}T(s)f(x)\,ds = \frac{1}{t}\int_0^t \mathcal{A}(T(s)f)(x)\,ds \\ &= \frac{1}{t}\int_0^t \Delta T(s)f(x) + b(x)\mathrm{grad}\, T(s)f(x)\,ds \\ &= \frac{1}{t}\int_0^t \Big(\big(\Delta T(s)f(x) - \Delta f(x)\big) + b(x)\big(\mathrm{grad}\, T(s)f(x) - \mathrm{grad}\, f(x)\big) \Big)\,ds + \Delta f(x) + b(x)\mathrm{grad}\, f(x) \\ &= \frac{1}{t}\int_0^t \Big(\big(\Delta T(s)f(x) - \Delta f(x)\big) + b(x)\big(\mathrm{grad}\, T(s)f(x) - \mathrm{grad}\, f(x)\big) \Big)\,ds + \mathcal{A}f(x). \end{aligned}$$

Since $\Delta T(s)f(x) \to \Delta f(x)$ and $\mathrm{grad}\, T(s)f(x) \to \mathrm{grad}\, f(x)$ as $s \to 0$ we obtain

$$\frac{1}{t}\big(T(t)f(x) - f(x)\big) \to \mathcal{A}f(x) \quad (t \to 0).$$

Clearly, $\mathcal{A}f \in C_b(\mathbb{R}^d)$.
Thus, it remains to show that

$$\sup_{t>0} \frac{\| T(t)f - f \|_\infty}{t} < \infty. \tag{1}$$

Fix $f \in C_c^\infty(\mathbb{R}^d)$ and let $n_0 \in \mathbb{N}$ such that $f \in C(\overline{B(0,n_0)})$. As in Theorem 14.1.1, let us consider the equation on the spatial domain $\overline{B(0,n)}$ for $n > n_0$,

$$\begin{cases} D_t u_n(t,x) &= \mathcal{A}u_n(t,x) & (t,x) \in (0,\infty) \times \overline{B(0,n)} \\ u_n(t,x) &= 0 & (t,x) \in (0,\infty) \times \partial B(0,n) \\ u_n(0,x) &= f(x) & x \in \overline{B(0,n)}. \end{cases} \tag{2}$$

By Lecture 12, we know that (2) has a unique bounded classical solution. From that, it is straight-forward to show that u_n is given by a bounded semigroup $T_n(\cdot)$ on $C(\overline{B(0,n)})$ through $u_n = T_n(\cdot)f$. It can be shown that $T_n(\cdot)$ is even an analytic semigroup (see also Remark 9.0.2) and that its generator A_n is an extension of $\mathcal{A}$ (if the latter is considered on $C(\overline{B(0,n)})$). Note that since $f \in C_c^\infty(\mathbb{R}^d)$, we have $f \in D(A_n)$. Hence, for $x \in \mathbb{R}^d$ and $t > 0$ we obtain

$$\big(T_n(t)f(x) - f(x)\big) = \int_0^t \frac{d}{ds}T_n(s)f(x)\,ds = \int_0^t A_n T_n(s)f(x)\,ds.$$

Moreover $A_n T_n(s) f = T_n(s) A_n f$, which follows by elementary properties of analytic semigroups. Furthermore, $A_n f = \mathcal{A} f$ and thus

$$\big(T_n(t) f(x) - f(x)\big) = \int_0^t T_n(s) \mathcal{A} f(x)\, ds.$$

By the proof of Theorem 14.1.1, step 1, it follows that for any $x \in \mathbb{R}^d$ and $t > 0$,

$$\lim_{n\to\infty} T_n(t) f(x) = \lim_{n\to\infty} u_n(t,x) = u(t,x) = T(t) f(x).$$

Since $\|T_n(s)\mathcal{A}f\|_{C(\overline{B(0,n)})} \leq \|\mathcal{A}f\|_{C(\overline{B(0,n)})}$ for all $s > 0$ and $n > n_0$, by dominated convergence we obtain

$$\big(T(t) f(x) - f(x)\big) = \int_0^t T(s) \mathcal{A} f(x)\, ds.$$

Using that $\|T(s)\mathcal{A}f\|_\infty \leq \|\mathcal{A}f\|_\infty$ for all $s > 0$, (1) follows.

We would like to remark that a similar argument as above is used in [2] to show that $T(\cdot)$ is strongly continuous on $C_0(\mathbb{R}^d)$.

Exercise 2

Fix $h \in \mathbb{R}^d \setminus \{0\}$ and consider the family of operators $\{S(t) : t \geq 0\}$ defined on $C_0(\mathbb{R}^d)$ by

$$(S(t)f)(x) = f(x + th), \quad t \geq 0,\ x \in \mathbb{R}^d.$$

Prove that $\{S(t)\}$ is a C_0-semigroup on $C_0(\mathbb{R}^d)$ whose generator is given by

$$D(B) = \{f \in C_0(\mathbb{R}^d) : D_h f \text{ exists and } D_h f \in C_0(\mathbb{R}^d)\}, \quad Bf = D_h f$$

and $i\mathbb{R} \subset \sigma(B)$.
(Note that we changed the notation of the directional derivative from $\langle h, Df\rangle$ to $D_h f$ to stress out that the directional derivative could exist, even if not all partial derivatives exist.)

We devide the proof in three steps.
Step 1: We show that S is a C_0-semigroup on $C_0(\mathbb{R}^d)$.
Let $f \in C_0(\mathbb{R}^d)$ and, as usual, we use the sup-norm on the space $C_0(\mathbb{R}^d)$. The operators $S(t)$ are clearly linear and since $\|S(t)f\|_\infty = \|f(\cdot + th)\|_\infty = \|f\|_\infty$ they are also bounded. Let $f \in C_0(\mathbb{R}^d)$. We have $S(0)f = f(\cdot + 0) = f$ and

$$S(t+s)f = f(\cdot + (t+s)h) = f(\cdot + th + sh) = S(t)S(s)f.$$

Hence the family satisfies the semigroup property.
For the strong continuity we use that every function in $C_0(\mathbb{R}^d)$ is uniformly continuous. So we have

$$\|S(t)f - f\| = \sup_{x\in\mathbb{R}^d} |f(x+th) - f(x)| \to 0, \quad \text{for } t \to 0.$$

Therefore S is a C_0-semigroup.
Step 2: We show that the generator B of S is given by the operator D_h with domain

$$\{f \in C_0(\mathbb{R}^d) : D_h f \text{ exists and } D_h f \in C_0(\mathbb{R}^d)\}.$$

Let us first assume that $f \in C_0(\mathbb{R}^d)$, $D_h f(x)$ exists for all $x \in \mathbb{R}^d$ and $D_h f \in C_0(\mathbb{R}^d)$. By definition of the directional derivative we have

$$D_h f(x + sh) = \lim_{t\to 0} \frac{f(x + (s+t)h) - f(x + sh)}{t} = \frac{d}{dt} f(x + th)\Big|_{t=s}.$$

Since $D_h f \in C_0(\mathbb{R}^d)$ is uniformly continuous we get for $t \to 0^+$

$$\begin{aligned}\left|\frac{f(x+th) - f(x)}{t} - D_h f(x)\right| &= \left|\frac{1}{t}\int_0^t D_h f(x+sh)\ ds - D_h f(x)\right| \\ &\leq \frac{1}{t}\int_0^t |D_h f(x+sh) - D_h f(x)| ds \to 0\end{aligned}$$

[2]Proposition 4.3 in *Metafune, Pallara, Wacker. Feller Semigroups on R^N, Semigroup Forum 65, 159–205, 2002*

uniformly in x. Therefore $f \in D(B)$ and $Bf = D_h f$.
Now let $f \in D(B)$. By definition of $D(B)$ we get that the right directional derivative

$$D_h^+ f(x) = \lim_{t \to 0^+} \frac{f(x+th) - f(x)}{t} = Bf(x)$$

exists for all $x \in \mathbb{R}^d$ and since $Bf \in C_0(\mathbb{R}^d)$ we get that the right directional derivative is continuous. This already implies that f is directional differentiable.
Indeed let $\xi \in \mathbb{R}^d$, $k := D_h^+ f(\xi)$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$D_h^+ f(x) \in (k - \varepsilon, k + \varepsilon) \quad \text{for } \|x - \xi\|_2 < \delta.$$

We will see that

$$k - \varepsilon < \frac{f(\xi + th) - f(\xi)}{t} < k + \varepsilon \quad \text{for } t \in \mathbb{R}, |t| \text{ small enough.} \tag{3}$$

Hence f is directional differentiable at ξ with $D_h f(\xi) = D_h^+ f(\xi)$.
To proof (3) we consider the functions

$$g_1(x) := f(x) - \frac{k - \varepsilon}{\|h\|_2^2} \langle h, x \rangle \quad \text{and} \quad g_2(x) := f(x) - \frac{k + \varepsilon}{\|h\|_2^2} \langle h, x \rangle, \quad x \in \mathbb{R}^d,$$

for which we get $D_h^+ g_1(x) > 0$ and $D_h^+ g_2(x) < 0$ for $\|x - \xi\|_2 < \delta$. This implies that g_1 is strictly increasing and g_2 is strictly decreasing in the direction of h, i.e. $0 < g_1(x + th) - g_1(x)$ and $0 > g_2(x + th) - g_2(x)$ for $t > 0$ small enough and all $\|x - \xi\|_2 < \delta$.
If we now use the definition of g_1 and g_2 and put $x = \xi$ or $x = \xi - th$ with $0 < t < \frac{\delta}{\|h\|}$ we get (3).
Step 3: We will show that for each $a \in \mathbb{R}$, ia is in the approximate point spectrum of B and therefore in the spectrum $\sigma(B)$.
For this it is sufficient to show that there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $D(B)$ such that $\|f_n\|_\infty = 1$ and $\lim_{n \to \infty} \|Bf_n - iaf_n\|_\infty = 0$. For $n \in \mathbb{N}$ let $f_n : \mathbb{R}^d \to \mathbb{C}$ be defined by

$$f_n(x) := e^{\frac{ia}{\|h\|_2^2} \langle h, x \rangle} \cdot e^{\frac{-\|x\|_2^2}{n}}, \quad x \in \mathbb{R}^d.$$

The functions f_n are in $C_0(\mathbb{R}^d)$, they are continuously differentiable and they satisfy $\|f_n\|_\infty = 1$. Now let us compute $D_h f_n = \langle h, Df_n \rangle$. We have

$$\frac{\partial f_n}{\partial x_j}(x) = \left(\frac{ia}{\|h\|_2^2} h_j - \frac{2x_j}{n} \right) e^{\frac{ia}{\|h\|_2^2} \langle h, x \rangle} e^{\frac{-\|x\|_2^2}{n}}, \quad \text{for } j = 1, ..., d$$

and hence

$$D_h f_n(x) = \langle h, Df_n(x) \rangle = \left(ia - \frac{2}{n} \langle h, x \rangle \right) e^{\frac{ia}{\|h\|_2^2} \langle h, x \rangle} e^{\frac{-\|x\|_2^2}{n}}.$$

We see that $D_h f_n \in C_0(\mathbb{R}^d)$ and therefore $f_n \in D(B)$ for all $n \in \mathbb{N}$.
Futhermore

$$\begin{aligned}
\|Bf_n - iaf_n\| &= \|\frac{2}{n} \langle h, x \rangle e^{\frac{ia}{\|h\|_2^2} \langle h, x \rangle} e^{\frac{-\|x\|_2^2}{n}} \| \\
&\leq \frac{2}{n} \sup_{x \in \mathbb{R}^d} \left| \langle h, x \rangle e^{\frac{-\|x\|_2^2}{n}} \right| = \frac{2}{n} \sqrt{\frac{n}{2e}} \|h\|_2 = \sqrt{\frac{2}{ne}} \|h\|_2
\end{aligned}$$

and hence $\lim_{n \to \infty} \|Bf_n - iaf_n\| = 0$.
Thus for each $a \in \mathbb{R}$ we have found a sequence $(f_n)_{n \in \mathbb{N}}$ in $D(B)$ with the above properties and therefore $ia \in \sigma(B)$.

Exercise 3

Let A be a linear operator with domain $D(A)$ on a Banach space X. Assume that $\rho(A) \neq \emptyset$. Prove that

$$\sigma(R(\lambda, A)) \setminus \{0\} = \left\{ \frac{1}{\lambda - \mu} : \mu \in \sigma(A) \right\}$$

and deduce that

$$\operatorname{dist}(\lambda, \sigma(A)) = \frac{1}{r(R(\lambda, A))} \geq \frac{1}{\|R(\lambda, A)\|_{L(X)}}$$

for $\lambda \in \rho(A)$, where $r(R(\lambda, A))$ denotes the spectral radius of $R(\lambda, A)$.

Let

$$\nu \in \sigma(R(\lambda, A)) \setminus \{0\}$$

be arbitrary. For λ fixed, we can express it as $\nu = \frac{1}{\lambda-\mu}$. This is no restriction since every complex number can be expressed in this way. Hence,

$$\frac{1}{\lambda-\mu}\mathrm{I} - R(\lambda, A)$$

is not bijective. Since we have

$$\frac{1}{\lambda-\mu}\mathrm{I} - R(\lambda, A) = \frac{1}{\lambda-\mu}\big((\lambda I - A) - (\lambda-\mu)I\big)R(\lambda, A) = \frac{(\mu\mathrm{I} - A)R(\lambda, A)}{\lambda-\mu},$$

the right side of the above equation can not be bijective. Since $R(\lambda, A)$ is a resolvent operator, it is bijective (from X to $D(A)$). Therefore, $(\mu\mathrm{I} - A)$ cannot be bijective (from $D(A)$ to X) and $\mu \in \sigma(A)$ follows immediately.
The other inclusion follows analogously.

For the second part we have

$$\operatorname{dist}(\lambda, \sigma(A)) = \inf_{\mu\in\sigma(A)} |\lambda-\mu| = \frac{1}{\sup_{\mu\in\sigma(A)} \frac{1}{|\lambda-\mu|}}.$$

With the previous result we obtain

$$\frac{1}{\sup_{\mu\in\sigma(A)} \frac{1}{|\lambda-\mu|}} = \frac{1}{\sup_{\psi\in\sigma(R(\lambda,A))\setminus\{0\}} |\psi|} = \frac{1}{r(R(\lambda, A))}.$$

Using the definition of the spectral radius and the submultiplicativity of the operator norm we further obtain

$$r(R(\lambda, A)) = \lim_{n\to\infty} \|R(\lambda, A)^n\|^{\frac{1}{n}} \le \|R(\lambda, A)\|.$$

Exercise 4

Consider the Laplacian Δ on $L^1(\mathbb{R}^d)$ with domain

$$D(\Delta) = \{u \in L^1(\mathbb{R}^d) \,:\, \Delta u \in L^1(\mathbb{R}^d)\},$$

where Δu is meant in the sense of distributions. Prove that $C_c^\infty(\mathbb{R}^d)$ is a core for Δ, i.e., $C_c^\infty(\mathbb{R}^d)$ is dense in $D(\Delta)$ with respect to the graph norm:

$$\|f\|_{D(\Delta)} = \|f\|_{L^1(\mathbb{R}^d)} + \|\Delta f\|_{L^1(\mathbb{R}^d)}, \qquad f \in D(\Delta).$$

Note: In the proof we use some generic constants which may change through the lines.
We divide the proof into two steps. In the first we argue that if $u \in D(\Delta)$, then u lies in $\mathrm{W}^{1,1}(\mathbb{R}^d)$. In the second we will provide a proof for the density.
Step 1. Here we have only to verify the following lemma:

Lemma 2. *It holds that $D(\Delta) \subset W^{1,1}(\mathbb{R}^d)$ and there exists $C(d)$ such that for all $u \in D(\Delta)$,*

$$\|\nabla u\|_{L^1(\mathbb{R}^d)} \le C(d)\left(\|u\|_{L^1(\mathbb{R}^d)} + \|\Delta u\|_{L^1(\mathbb{R}^d)}\right).$$

As indicated in (15.14) in Lecture 15, this result can be found in *H. Tanabe,* Functional Analytic Methods for Partial Differential Equations*, Dekker, 1997.*, Theorem 5.8. However, we also provide a proof in the appendix of this note.
Step 2. For the proof that $C_c^\infty(\mathbb{R}^d)$ is dense in $D(\Delta)$ we choose $\varphi \in C_c^\infty(\Omega)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_1(0)$, $\varphi \equiv 0$ on $\mathbb{R}^d \setminus B_2(0)$. Further, for $R > 0$ we define $\varphi_R(x) = \varphi(\frac{x}{R})$ ($x \in \mathbb{R}^d$). For a given $u \in D(\Delta)$ we look at the function $\varphi_R u_\varepsilon$ where u_ε is again the mollification of u with the standard mollifier. Obviously $\varphi_R u_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ for all $\varepsilon > 0$ and $R > 0$.
For a given $\gamma > 0$ we can find $\varepsilon_0 > 0$ and $R_0 > 1$ with

$$\|u - \varphi_R u_\varepsilon\|_{L^1(\mathbb{R}^d)} \le \|u - u_\varepsilon\|_{L^1(\mathbb{R}^d)} + \|u_\varepsilon - \varphi_R u_\varepsilon\|_{L^1(\mathbb{R}^d)} \le \frac{\gamma}{2}$$

for $R > R_0$ and $0 < \varepsilon < \varepsilon_0$. Let $M := \sup_{\varepsilon>0} \|u_\varepsilon\|_{W^{1,1}(\mathbb{R}^d)} = \|u\|_{W^{1,1}(\mathbb{R}^d)} < \infty$. Then

$$\begin{aligned}
\|\Delta u - \Delta(\varphi_R u_\varepsilon)\|_{L^1(\mathbb{R}^d)} &\le \|(\Delta\varphi_R)u_\varepsilon\|_{L^1(\mathbb{R}^d)} + 2\|\nabla\varphi_R \cdot \nabla u_\varepsilon\|_{L^1(\mathbb{R}^d)} + \|\Delta u - \varphi_R(\Delta u_\varepsilon)\|_{L^1(\mathbb{R}^d)} \\
&\le \frac{\|\Delta\varphi\|_\infty}{R^2} M + \frac{2}{R}\|\nabla\varphi\|_\infty M + \|\Delta u - \varphi_R(\Delta u)_\varepsilon\|_{L^1(\mathbb{R}^d)} \\
&\le \frac{C(\varphi, M)}{R} + \|\Delta u - (\Delta u)_\varepsilon\|_{L^1(\mathbb{R}^d)} + \|(\Delta u)_\varepsilon - \varphi_R(\Delta u)_\varepsilon\|_{L^1(\mathbb{R}^d)} =: I
\end{aligned}$$

we can choose $\varepsilon_1 > 0$ and $R_1 > 1$ such that $I \leq \frac{\gamma}{2}$ if $0 < \varepsilon < \varepsilon_1$ and $R > R_1$.
Choosing $0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ and $R > \max\{R_0, R_1\}$ we obtain

$$\|u - \varphi_R u_\varepsilon\|_\Delta \leq \gamma$$

which concludes the proof.

Appendix - a proof of Lemma 2

Lemma 3. *For $x_0 \in \mathbb{R}^d$ and $u \in W^{2,\infty}(\mathbb{R}^d)$ the following inequality holds*

$$\|\nabla u\|_{L^1(B_1(x_0))} \leq C(d) \left(\|u\|_{L^1(B_2(x_0))} + \|\Delta u\|_{L^1(B_2(x_0))}\right)$$

Proof. We set

$$f = \begin{cases} \Delta u & \text{on } B_2(x_0) \\ 0 & \text{else} \end{cases} \in L^1_c(\mathbb{R}^d),$$

then the Newtonian potential w of f is in $C^1(\mathbb{R}^d)$ with $\nabla w = C(d) \int\limits_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} f(y)\, dy$ and $\Delta w = f$. Now we have with Fubini's Theorem

$$\begin{aligned}
\|\nabla w\|_{L^1(B_2(x_0))} &\leq C(d) \int\limits_{B_2(x_0)} \int\limits_{\mathbb{R}^d} |x-y|^{1-d} |f(y)|\, dy\, dx \\
&\leq C(d) \int\limits_{\mathbb{R}^d} \underbrace{\int\limits_{B_2(x_0)} |x-y|^{1-d}\, dx}_{\leq \int\limits_{B_2(0)} |\zeta|^{1-d}\, d\zeta \leq C(d)} |f(y)|\, dy \\
&\leq C(d) \|\Delta u\|_{L^1(B_2(x_0))}.
\end{aligned}$$

Using the fact that $u - w$ is harmonic on $B_2(x_0)$ and with the Caccioppoli inequality[3] and the Poincaré inequality[4] we get

$$\begin{aligned}
\|\nabla u\|_{L^1(B_1(x_0))} &\leq \|\nabla(u-w)\|_{L^1(B_1(x_0))} + \|\nabla w\|_{L^1(B_1(x_0))} \\
&\leq C(d)\|u - w + \bar{w}\|_{L^1(B_2(x_0))} + \|\nabla w\|_{L^1(B_1(x_0))} \\
&\leq C(d) \left(\|u\|_{L^1(B_2(x_0))} + \|w - \bar{w}\|_{L^1(B_2(x_0))}\right) + \|\nabla w\|_{L^1(B_2(x_0))} \\
&\leq C(d) \left(\|u\|_{L^1(B_2(x_0))} + \|\nabla w\|_{L^1(B_2(x_0))}\right).
\end{aligned}$$

With $\bar{w}$ we denote the integral average of w on $B_2(x_0)$. The both estimates combined prove the lemma. □

To prove that for $u \in D(\Delta)$ we already have $u \in W^{1,1}(\mathbb{R}^d)$, we mollify u with the standard mollifier and get $u_\varepsilon \to u$ and $\Delta u_\varepsilon \to u$ in $L^1(\mathbb{R}^d)$ for $\varepsilon \to 0$. For u_ε and $u_\varepsilon - u_\delta$ we can use the estimates of the lemma. Now we can cover $\mathbb{R}^d$ with countable many balls $B_1(x_i)$ such that $\sum\limits_{i \in \mathbb{N}} \mathbb{1}_{B_1(x_i)}(x) \leq C(d)$ ($\mathbb{1}_A$ denotes the characteristic function of A). By summation of the estimates on those balls we get

$$\|\nabla u_\varepsilon\|_{L^1(\mathbb{R}^d)} \leq C(d) \left(\|u_\varepsilon\|_{L^1(\mathbb{R}^d)} + \|\Delta u_\varepsilon\|_{L^1(\mathbb{R}^d)}\right)$$

and

$$\|\nabla(u_\varepsilon - u_\delta)\|_{L^1(\mathbb{R}^d)} \leq C(d) \left(\|u_\varepsilon - u_\delta\|_{L^1(\mathbb{R}^d)} + \|\Delta(u_\varepsilon - u_\delta)\|_{L^1(\mathbb{R}^d)}\right).$$

The last inequality gives us a Cauchy property for $(\nabla u_\varepsilon)_{\varepsilon > 0}$ in $L^1(\mathbb{R}^d)$, which implies $\nabla u_\varepsilon \to \nabla u$ in $L^1(\mathbb{R}^d)$ and we get $u \in W^{1,1}(\mathbb{R}^d)$ with

$$\|\nabla u\|_{L^1(\mathbb{R}^d)} \leq C(d) \left(\|u\|_{L^1(\mathbb{R}^d)} + \|\Delta u\|_{L^1(\mathbb{R}^d)}\right).$$

[3] Let $f \in W^{1,1}(\mathbb{R}^d)$ be harmonic and $\lambda \in \mathbb{R}$. By the mean value property of harmonic function we get the estimate $|Df(x)| = |D(f(x) - \lambda)| \leq C(d) \int\limits_{B_1(x)} |f(x) - \lambda|\, dx$. From that we deduce

$$\|Df\|_{L^1(B_1(0))} \leq C(d) \int_{B_1(0)} \int_{B_1(y)} |f(x) - \lambda|\, dx\, dy \leq C(d) \int_{B_1(0)} \int_{B_2(0)} |f(x) - \lambda|\, dx\, dy = C(d)\|f - \lambda\|_{L^1(B_2(0))}.$$

Inequalities of the form $\|Df\| \leq C\|f - \lambda\|$ are often called Caccioppoli inequalities and the standard version is for $W^{1,2}$ functions.

[4] This version of Poincaré's inequality is sometimes also referred to as "Poincaré-Wirtinger inequality"