Solutions of the Exercises of Lecture 1

Salerno Team

Exercise 1. *Let* $B = (b_{ij})_{1\le i,j\le d}, C = (c_{ij})_{1\le i,j\le d}$ *be two real and symmetric matrices. Assume that* B *is positive semi-definite and* C *is negative semi-definite. Prove that*

$$Tr(BC) = \sum_{i,j=1}^{d} b_{ij}c_{ij} \le 0.$$

Proof. Since C is real and symmetric, there exists an orthogonal basis $\{\phi_1, ..., \phi_d\}$ of $\mathbb{R}^d$, constituted of eigenvectors of C, i.e.,

$$C\phi_i = \lambda_i\phi_i, \qquad \lambda_i \le 0, \quad i = 1, \dots, d.$$

Let M be the orthogonal matrix which has as columns the vectors ϕ_i. One knows that the trace does not depend on the chosen basis, hence, one can write the trace of BC as follows

$$Tr(BC) = Tr(M^T BCM) = \sum_{i=1}^{d}\langle BC\phi_i, \phi_i\rangle = \sum_{i=1}^{d}\langle B\lambda_i\phi_i, \phi_i\rangle = \sum_{i=1}^{d}\lambda_i\langle B\phi_i, \phi_i\rangle \le 0.$$

This ends the proof. □

Exercise 2. *Consider the function* $u(t, x) := 1 - x^2 - 2t$ *for* $(t, x) \in \Omega := (0, T] \times (0, 1)$.

(*a*) *Verify that* u *is a solution to the heat equation* $D_t u(t, x) = \Delta\, u(t, x)$.

(*b*) *Find the minimum and the maximum of* u *on the closed rectangle* $\overline{\Omega_T} := [0, T] \times [0, 1]$ *for a fixed* $T > 0$ *without using the maximum principle.*

(*c*) *Find the minimum and the maximum of* u *on* $\overline{\Omega_T}$ *by using the weak maximum principle.*

Proof. (a) It suffices to notice that $D_t u(t, x) = -2$ and $\Delta\, u(t, x) = -2$.

(b) Observe that in $\overline{\Omega_T}$

$$-2T \le 1 - x^2 - 2t \le 1,$$

and $u(T, 1) = -2T,\ u(0, 0) = 1$. So, we have $\max_{\overline{\Omega_T}} u = 1$ and $\min_{\overline{\Omega_T}} u = -2T$.

(c) From (a) we know that

$$D_t u(t, x) = \Delta\, u(t, x) \Rightarrow D_t u(t, x) - \Delta\, u(t, x) = 0.$$

The parabolic boundary of Ω_T is $\Gamma_T = (\{0\} \times [0, 1]) \cup ([0, T] \times \{0, 1\})$, so by the weak maximum principle

$$\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u.$$

Hence

$$\max_{\Gamma_T} u = \max_{0\le x\le 1}(1-x^2)\vee \max_{0\le t\le T}(1-2t)\vee \max_{0\le t\le T}(-2t) = 1.$$

By the same argument we find the minimum

$$\min_{\Gamma_T} u = -2T.$$

□

Exercise 3. *Let $u \in C^{1,2}((0,+\infty)\times(-\frac{\pi}{2},\frac{\pi}{2}))\cap C([0,+\infty)\times[-\frac{\pi}{2},\frac{\pi}{2}])$ satisfy the inequality $D_t u(x) - \Delta\, u(x) \ge \cos x$ on $(0,+\infty)\times(-\frac{\pi}{2},\frac{\pi}{2})$. Moreover assume that: $u(t,-\frac{\pi}{2}) \ge 0$, $u(t,\frac{\pi}{2}) \ge 0$ for all $t > 0$ and $u(0,x) \ge 2\cos x$ for all $x \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Show that*

$$u(t,x) \ge (1+e^{-t})\cos x, \qquad (t,x)\in[0,+\infty)\times[-\frac{\pi}{2},\frac{\pi}{2}]. \tag{1}$$

Proof. Consider the function

$$v(t,x) = u(t,x) - (1+e^{-t})\cos x \tag{2}$$

then we have

$$\begin{aligned} D_t v(t,x) - \Delta v(t,x) &= D_t u(t,x) - \Delta\, u(t,x) + e^{-t}\cos x - (1+e^{-t})\cos x \\ &= D_t u(t,x) - \Delta\, u(t,x) - \cos x \ge 0 \end{aligned}$$

Moreover, $v(0,x) = u(0,x) - 2\cos x \ge 0$ for all $x \in [-\frac{\pi}{2},\frac{\pi}{2}]$ and $v(t,\pm\frac{\pi}{2}) = u(t,\pm\frac{\pi}{2}) \ge 0$ for all $t > 0$. Then setting $\Omega_T = (0,T]\times]-\frac{\pi}{2},\frac{\pi}{2}[$ we have $v \ge 0$ on Γ_T for all $T > 0$. Hence, by comparison principle (Corollary 1.1.5 see also Remark 1.1.6), we have $v \ge 0$ in Ω_T for all $T > 0$. Then, (1) holds in $[0,+\infty)\times[-\frac{\pi}{2},\frac{\pi}{2}]$. □

Exercise 4. *Let $u \in C^{1,2}((0,T]\times\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)$ be a solution to the heat equation*

$$\begin{cases} D_t u(t,x) = \Delta\, u(t,x), & (t,x)\in(0,T]\times\mathbb{R}^d, \\ u(0,x) = g(x), & x\in\mathbb{R}^d \end{cases} \tag{3}$$

satisfying

$$u(t,x) \le M e^{a|x|^2}, \qquad \forall (t,x)\in[0,T]\times\mathbb{R}^d$$

for some constants $M, a \ge 0$, where $g \in C(\mathbb{R}^d)$.

(a) Assume first that $4aT < 1$ which implies that, there exists $\varepsilon > 0$ such that $4a(T+\varepsilon) < 1$. Fix $\nu > 0$ and consider the function

$$v(t,x) = u(t,x) - \frac{\nu}{(T+\varepsilon-t)^{\frac{d}{2}}}\exp\left(\frac{|x|^2}{4(T+\varepsilon-t)}\right), \quad (t,x)\in[0,T]\times\mathbb{R}^d.$$

(i) Prove that v solves $D_t v - \Delta v = 0$ on $(0,T]\times\mathbb{R}^d$ and $v \in C([0,T]\times\mathbb{R}^d)$.

(ii) By applying the weak maximum principle to the function v on the cylinder $[0,T]\times B_r(0)$, show that

$$\max_{[0,T]\times\overline{B_r(0)}} v \le \sup_{\mathbb{R}^d} g$$

for sufficiently large r.

(iii) By letting $\nu \to 0$, deduce that

$$\sup_{[0,T]\times\mathbb{R}^d} u = \sup_{\mathbb{R}^d} g. \tag{4}$$

(b) Prove (4) *without the assumption* $4aT < 1$.

Proof. (*a*) Assume that $4aT < 1$, let $\varepsilon > 0$ such that $4a(T+\varepsilon) < 1$ and consider the function

$$v(t,x) = u(t,x) - \frac{\nu}{(T+\varepsilon-t)^{\frac{d}{2}}} \exp\left(\frac{|x|^2}{4(T+\varepsilon-t)}\right), \quad (t,x) \in [0,T]\times\mathbb{R}^d$$

with $\nu > 0$ fixed.

(*i*) Let $w(t,x) := \frac{\nu}{(T+\varepsilon-t)^{\frac{d}{2}}} \exp\left(\frac{|x|^2}{4(T+\varepsilon-t)}\right)$ we have

$$D_t w(t,x) = \frac{w(t,x)}{(T+\varepsilon-t)}\left(\frac{d}{2} + \frac{|x|^2}{4(T+\varepsilon-t)}\right)$$

and

$$D_j w(t,x) = \frac{w(t,x)}{(T+\varepsilon-t)}\frac{x_j}{2} \text{ and } D_{jj} w(t,x) = \frac{w(t,x)}{(T+\varepsilon-t)}\left(\frac{1}{2} + \frac{x_j^2}{4(T+\varepsilon-t)}\right).$$

Then

$$\Delta w(t,x) = \frac{w(t,x)}{(T+\varepsilon-t)}\left(\frac{d}{2} + \frac{|x|^2}{4(T+\varepsilon-t)}\right).$$

So, $D_t w = \Delta w$ and consequently $D_t v = \Delta v$ on $(0,T]\times\mathbb{R}^d$. Moreover, we note that $w(t,x)$ is continuous on $[0,T]\times\mathbb{R}^d$ and then $v \in C\left([0,T]\times\mathbb{R}^d\right)$.

(*ii*) Set $\Omega_T = (0,T]\times B_r(0)$ and assume that $\sup_{\mathbb{R}^d} g < +\infty$. If not there is nothing to prove.
The function v is a solution of the problem

$$\begin{cases} D_t v(t,x) - \Delta\, v(t,x) = 0, & (t,x)\in\Omega_T \\ v(t,x) = u(t,x) - \frac{\nu}{(T+\varepsilon-t)^{\frac{d}{2}}}\exp\left(\frac{r^2}{4(T+\varepsilon-t)}\right), & (t,x)\in[0,T]\times\partial B_r(0) \\ v(0,x) = g(x) - \frac{\nu}{(T+\varepsilon)^{\frac{d}{2}}}\exp\left(\frac{|x|^2}{4(T+\varepsilon)}\right), & x\in B_r(0). \end{cases} \tag{5}$$

We have $v(0,x) \le g(x) \le \sup_{\mathbb{R}^d} g$ for all $x \in B_r(0)$.
Moreover, on $[0,T]\times\partial B_r(0)$ we have $|x| = r$ and so

$$\begin{aligned} v(t,x) &\le Me^{ar^2} - \frac{\nu}{(T+\varepsilon-t)^{\frac{d}{2}}}\exp\left(\frac{r^2}{4(T+\varepsilon-t)}\right) \\ &\le Me^{ar^2} - \frac{\nu}{(T+\varepsilon)^{\frac{d}{2}}}\exp\left(\frac{r^2}{4(T+\varepsilon)}\right) \\ &= e^{ar^2}\left[M - \frac{\nu}{(T+\varepsilon)^{\frac{d}{2}}}\exp\left(\left(\frac{1}{4(T+\varepsilon)} - a\right)r^2\right)\right]. \end{aligned}$$

Since $a < \frac{1}{4(T+\varepsilon)}$, the right hand side tends to $-\infty$ as r goes to $+\infty$, and therefore

$$v(t,x) \le \sup_{\mathbb{R}^d} g \quad \text{on } [0,T] \times \partial B_r(0)$$

for r sufficiently large. Thus, for large r, we have $v(t,x) \le \sup_{\mathbb{R}^d} g$ for all $(t,x) \in \Gamma_T$. Hence, by the weak maximum principle, we obtain

$$\max_{[0,T]\times\overline{B_r(0)}} v \le \sup_{\mathbb{R}^d} g$$

for sufficiently large r, and hence for all $r > 0$.

(*iii*) By (ii) we have $v(t,x) \le \sup_{\mathbb{R}^d} g$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Then, by letting $\nu \to 0$, we obtain $u(t,x) \le \sup_{\mathbb{R}^d} g$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$. On the other hand,

$$\sup_{\mathbb{R}^d} g = \sup_{x\in\mathbb{R}^d} u(0,x) \le \sup_{[0,T]\times\mathbb{R}^d} u(t,x).$$

Thus,

$$\sup_{[0,T]\times\mathbb{R}^d} u = \sup_{\mathbb{R}^d} g.$$

(*b*) We choose $n \in \mathbb{N}$ such that $\frac{4aT}{n} < 1$. Then we split the interval $[0,T]$ in n intervals $[t_k, t_{k+1}]$, $k = 0, \dots, (n-1)$ such that $t_0 = 0$, $t_1 = \frac{T}{n}$ and $t_k = kt_1$, $k = 2, \dots, n$. In the interval $[0,t_1]$, since $4at_1 < 1$, thanks to (*i*) we get (4). In order to obtain the same result in $[t_1,t_2]$ we consider the function $z_1(t,x) = u(t+t_1,x)$ for $t \in [0,t_1]$, $x \in \mathbb{R}^d$. Then $z_1 \in C^{1,2}((0,t_1] \times \mathbb{R}^d) \cap C([0,t_1] \times \mathbb{R}^d)$ and solves

$$\begin{cases} D_t z_1(t,x) = \Delta\, z_1(t,x), & (t,x) \in (0,t_1] \times \mathbb{R}^d, \\ z_1(0,x) = u(t_1,x), & x \in \mathbb{R}^d. \end{cases}$$

As before, one obtains $\sup_{(t,x)\in[0,t_1]\times\mathbb{R}^d} z_1(t,x) = \sup_{x\in\mathbb{R}^d} u(t_1,x)$. In order to prove (4), we notice that

$$\sup_{(t,x)\in[t_1,t_2]\times\mathbb{R}^d} u(t,x) = \sup_{(t,x)\in[0,t_1]\times\mathbb{R}^d} z_1(t,x) = \sup_{x\in\mathbb{R}^d} u(t_1,x) \le \sup_{(\mathbb{R}^d} g.$$

Then $\sup_{[0,t_2]\times\mathbb{R}^d} u = \sup_{\mathbb{R}^d} g$. Continuing in this way we obtain (4) for every $T > 0$.

□

Exercise 5. *Assume that* $u_0 \in C(\overline{\Omega})$ *and* $u \in C^{1,2}((0,T] \times \Omega) \cap C([0,T] \times \overline{\Omega})$ *is a solution to the heat equation with Dirichlet boundary conditions*

$$\begin{cases} D_t u(t,x) = \Delta u(t,x), & (t,x) \in (0,T] \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases}$$

Consider the function $\Phi : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\Phi(s) := \begin{cases} 1 - e^{-s^2}, & s \ge 0, \\ 0, & s < 0. \end{cases} \tag{6}$$

Set

$$H(\tau) := \begin{cases} \int_0^\tau \Phi(s)\, ds, & \tau \geq 0, \\ 0, & \tau < 0, \end{cases} \quad \textit{and } \varphi(t) := \int_\Omega H(u(t,x) - K)\, dx,\ t \in [0,T]$$

and $K := \max\left(\sup_{x\in\Omega} u_0(x), 0\right)$.

(a) Prove that Φ *is a* C^1 *-function, increasing and its derivative is bounded by 1.*

(b) Prove that $\varphi(0) = 0$, $\varphi \geq 0$ *on* $[0,T]$ *and* $\varphi \in C^1((0,T],\mathbb{R}) \cap C([0,T],\mathbb{R})$. *Compute* φ' *and deduce that*

$$u(t,x) \leq K, \quad \forall (t,x) \in [0,T] \times \overline{\Omega}.$$

Proof. (a) By (6), we can compute

$$\Phi'(s) = \begin{cases} 2se^{-s^2}, & s \geq 0, \\ 0, & s < 0 \end{cases}$$

and observe that $\Phi'(s) \geq 0$ for all $s \in \mathbb{R}$ and $\Phi'(s) \leq 1$ since $2s \leq e^{s^2}$ holds for all $s \in \mathbb{R}$. Therefore, Φ is a C^1 -function, increasing and its derivative is bounded by 1.

(b) We observe that

$$\varphi(0) = \int_\Omega H(u_0(x) - K)\, dx = 0,$$

since $K \geq u_0(x)$ for all $x \in \Omega$.
First assume that $K > 0$. So,

$$\varphi(t) = \int_{\Omega'} H(u(t,x) - K)\, dx,\ t \in [0,T],$$

where $\Omega' := \cup_{t\in(0,T]}\Omega'_t$ with $\Omega'_t := \{x \in \Omega : u(t,x) - K > 0\}$ for $t \in (0,T]$. We observe that Ω' is an open subset of Ω and since $u(t,x) - K < 0$ on $(0,T] \times \partial\Omega$, we have $\overline{\Omega'} \subset \Omega$. Hence $u \in C^{1,2}((0,T] \times \overline{\Omega'})$ and so we can differentiate under the integral sign obtaining

$$\varphi'(t) = \int_{\Omega'} \Phi(u(t,x) - K) D_t u(t,x)\, dx.$$

By applying the Gauss-Green formula we obtain

$$\begin{aligned} \varphi'(t) &= \int_{\Omega'} \Phi(u(t,x) - K)\Delta u(t,x)\, dx \\ &= \int_{\partial\Omega'} \Phi(u(t,x) - K)\frac{\partial u}{\partial n}\, d\sigma - \int_{\Omega'} \nabla[\Phi(u(t,x) - K)] \cdot \nabla u\, dx \\ &= -\int_{\Omega'} \Phi'(u(t,x) - K)|\nabla u|^2\, dx \leq 0. \end{aligned}$$

Therefore, we have $\varphi \geq 0$, $\varphi(0) = 0$ and φ decreasing, which implies $\varphi(t) = 0$ for all $t \in [0, T]$. Thus since $H(t) \geq 0$ we have $H(u(t,x) - K) = 0$, which implies $u(t,x) - K \leq 0$ on $[0,T] \times \overline{\Omega}$.
For the more general case we consider instead of K the positive constant $K_\varepsilon := \max(\sup_{x\in\Omega} u_0(x), \varepsilon)$. So, the above arguments give us $u(t,x) \leq K_\varepsilon$ for all $(t,x) \in [0,T] \times \overline{\Omega}$. The result is now obtained by letting $\varepsilon \to 0^+$. □