Solutions to the Exercises of Lecture 3

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Exercise 1

(i) Let $R \gg r > 0$ and let γ_i for $i = 1, 2, 3$ be the three parts of $\gamma := \gamma_{r,\eta,0}$. Let $\tilde{\gamma}_4 : [\eta, 2\pi - \eta] \to \mathbb{C}$, $\theta \mapsto Re^{i\theta}$, $\tilde{\gamma}_2 = \gamma_2$, $\tilde{\gamma}_1 = \gamma_1|_{[r,R]}$ and $\tilde{\gamma}_3 = \gamma_3|_{[r,R]}$. Then the "union" of the curves $-\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3, \tilde{\gamma}_4$ is a closed curve $\tilde{\gamma}$. Since $\lambda \mapsto e^{t\lambda}$ is holomorphic, we have by the Cauchy integral formula that $\int_{\tilde{\gamma}} e^{t\lambda} d\lambda = 0$. Hence in order to prove $\int_\gamma e^{t\lambda} d\lambda = 0$ it suffices to show that the contributions of the integrals over $\tilde{\gamma}_4$ and the missing parts $\gamma_1|_{[R,\infty)}$ and $\gamma_3|_{[R,\infty)}$ converge to 0 for $R \to \infty$.
Firstly,

$$\left| \int\limits_{\tilde{\gamma}_4} e^{t\lambda} d\lambda \right| \leq 2\pi R \max_{\theta \in [\eta, 2\pi-\eta]} e^{tR\cos(\theta)} \overset{R\to\infty}{\longrightarrow} 0,$$

as there exists a $\delta < 0$ such that $\cos\theta < \delta$ for all $\theta \in [\eta, 2\pi - \eta]$. Secondly, for the missing part on γ_3 we have

$$\begin{aligned}
\left| \int\limits_{\gamma_3|_{[R,\infty)}} e^{t\lambda} d\lambda \right| &= \left| \int\limits_R^\infty e^{t\rho e^{i\eta}} e^{i\eta} d\rho \right| \\
&\leq \int\limits_R^\infty \left| e^{t \operatorname{Re}(\rho e^{i\eta}) + it \operatorname{Im}(\rho e^{i\eta})} \right| d\rho \\
&\leq \int\limits_R^\infty e^{t \operatorname{Re}(\rho e^{i\eta})} d\rho \\
&= \int\limits_R^\infty e^{t\rho\cos\eta} d\rho \\
&\overset{R\to\infty}{\longrightarrow} 0,
\end{aligned}$$

as $\cos\eta < 0$. The missing part on $-\gamma_1$ is treated in the same way. Thus, (3.8) follows.

(ii) First equation:
The argumentation is similar to (i). For $\gamma := \gamma_{2r,\eta',0}$ we use analogous curves as in (i) involving $R > 0$ and the closed curve $\tilde{\gamma}$. Let $\lambda \in \gamma_{r,\eta,0}$. We suppose in addition that $|\lambda| < \frac{R}{2}$. This ensures in particular that λ is in the interior of $\tilde{\gamma}$. So we have

$$\int_{\tilde{\gamma}} \frac{e^{s\mu}}{\mu - \lambda} d\mu = 2\pi i e^{s\lambda},$$

where we made use of Cauchy's integral theorem. Then

$$\begin{aligned}
\left|\int_{\tilde{\gamma}_4} \frac{e^{\mu s}}{\mu-\lambda} d\mu\right| &= \left|\int_{\eta'}^{2\pi-\eta'} \frac{e^{Re^{it}s}}{Re^{it}-\lambda} iRe^{it} dt\right| \\
&\leq \int_{\eta'}^{2\pi-\eta'} \frac{|e^{Re^{it}s}|}{|e^{it}-\frac{\lambda}{R}|} dt \\
&\leq \int_{\eta'}^{2\pi-\eta'} \frac{e^{R\cos(t)s}}{\frac{1}{2}} dt \\
&= 2\int_{\eta'}^{2\pi-\eta'} e^{Rs\cos(t)} dt \\
&\stackrel{R\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where we have used the dominated convergence theorem (note that $Rs > 0$ and that $\cos(t) < 0$ for all $t \in (\eta', 2\pi - \eta')$).

As in (i), it remains to consider the rays $\gamma_1|_{[R,\infty)}$ and $\gamma_3|_{[R,\infty)}$.

$$\begin{aligned}
\left|\int_{\gamma_3|_{[R,\infty)}} \frac{e^{\mu s}}{\mu-\lambda} d\mu\right| &= \left|\int_{R}^{\infty} \frac{e^{te^{i\eta'}s}}{te^{i\eta'}-\lambda} e^{i\eta'} dt\right| \\
&\leq \int_{R}^{\infty} \frac{e^{ts\cos(\eta')}}{\frac{R}{2}} dt \\
&= \frac{2e^{Rs\cos(\eta')}}{Rs\cos\eta'} \\
&\stackrel{R\to\infty}{\longrightarrow} 0,
\end{aligned}$$

since $\cos(\eta') < 0$.

The second equation is analogous to the first one, but easier since the only singularity μ is not in the interior of the constructed closed curve. So the same arguments yield that the integral is 0.

Exercise 2

Choose $\omega \in \mathbb{R}$, $\theta_0 \in (\frac{\pi}{2}, \pi)$ and $M \geq 0$ such that $\Sigma_{\omega,\theta_0} \subset \rho(A)$ and $\|R(\lambda, A)\|_{L(X)} \leq M|\lambda - \omega|^{-1}$ for every $\lambda \in \Sigma_{\omega,\theta_0}$.

(i) We prove that $Bx := Ax - \alpha x$ is sectorial by using Proposition 3.2.8: It is

$$\Pi := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\mathrm{Re}(\alpha) + 1 + \omega\} \subset \rho(B)$$

Also if $\lambda \in \Pi$ we have:

$$\|\lambda R(\lambda, B)\|_{L(X)} = \|\lambda R(\lambda + \alpha, A)\| \leq M\frac{|\lambda|}{|\lambda + \alpha - \omega|}$$

It is easy to see that $\frac{|\lambda|}{|\lambda+\alpha-\omega|}$ is bounded for all $\lambda \in \Pi$. It follows that B is sectorial by Proposition 3.2.8.

(ii) Define $S(t) := e^{-\alpha t}T(t)$. Now, using Theorem 3.2.2, we have

$$\frac{d}{dt}S(t) = -\alpha e^{-\alpha t}T(t) + e^{-\alpha t}\frac{d}{dt}T(t) = -\alpha e^{-\alpha t}T(t) + e^{-\alpha t}AT(t) = (A - \alpha I)S(t).$$

This shows that $\{S(t)\}$ is the associated semigroup.

(iii) Define $Cx := \alpha Ax$. For which α is C sectorial? Because (i) shows that C is sectorial iff $\alpha(A - \omega)$ is sectorial, we can assume $\omega = 0$. Take $\lambda \in \mathbb{C}$. Then $\lambda I - C$ is bijective if $\lambda\alpha^{-1} \in \Sigma_{0,\theta_0}$. Because $\arg(\lambda\alpha^{-1}) = \arg\lambda - \arg\alpha$ we have $\rho(C) \subset \Sigma_{0,\theta_0 - |\arg\alpha|}$ which is a suitable sector whenever $\theta_0 - |\arg\alpha| \in (\frac{\pi}{2}, \pi)$. Now take $\lambda \in \Sigma_{0,\theta_0 - |\arg\alpha|}$. It is

$$\|R(\lambda, C)\|_{L(X)} = \|\alpha R(\lambda\alpha^{-1}, A)\| \leq M|\alpha|^2|\lambda|^{-1}$$

So C is sectorial if $\theta_0 - |\arg\alpha| \in (\frac{\pi}{2}, \pi)$, i.e., if $\theta_0 - \frac{\pi}{2} > |\arg\alpha|$.

Exercise 3

(i) We have that

$$x = (\mu - \lambda)^{-1}(\mu x - \lambda x) = (\mu I - A)(\mu - \lambda)^{-1}x$$

from which the claim follows.

(ii) Since $x \in D(A)$ and $\lambda x \in D(A)$ it follows from Proposition 3.2.7 that

$$\frac{d}{dt}T(t)x = AT(t)x = \lambda T(t)x.$$

By standard ODE arguments we obtain that $T(t)x = e^{\lambda t}x$.

(iii) We prove the claim in three steps: 1^{st} *step*: By Proposition 3.2.7 (iii) the operator

$$\begin{aligned} \tilde{A} \colon D = \{x \in D(A), Ax \in D(A)\} &\to D(A) \\ x &\mapsto AX \end{aligned}$$

generates the semigroup $(\tilde{T}(t))_{t\geq 0}$ on $\overline{D(A)}$ where $\tilde{T} = T_{|\overline{D(A)}}$. The operator $\tilde{A}$ is sectorial since A is and hence we may replace X by $\overline{D(A)}$ and $D(A)$ by D to solve the exercise. It follows that we may assume without loss of generality that $D(A)$ is dense in X.

2^{nd} *step*: By Remark 2.1.17 we know that A generates a C_0-group $(T(t))_{t\in\mathbb{R}}$ which is continuous in operator norm on $\mathbb{R}\backslash\{0\}$ by Theorem 3.2.2 (iv). Using Theorem 3.2.2 (iii) with $k = 0$ we compute

$$\|I - T(t)\| \leq \|T(-1)\| \, \|T(1) - T(1+t)\| \leq M_0\|T(1) - T(1+t)\|$$

which converges to 0 as $t \to 0$. This shows that $t \mapsto T(t)$ is continuous in operator norm on $\mathbb{R}$.

3^{rd} *step*: Since $t \mapsto T(t)$ is continuous, Theorem A.2.4 implies that

$$\frac{1}{h}\int_t^{t+h} T(s)\,ds \to T(t)$$

in operator norm. Lemma 2.1.11 shows that $\frac{1}{h}\int_0^h T(t)\,dt$ is invertible if h is small enough. Fix such an h and compute

$$\begin{aligned}
\frac{T(t)-I}{t}\int_0^h T(s)\,ds &= \frac{1}{t}\left(\int_0^h T(t+s)\,ds - \int_0^h T(s)\,ds\right)\\
&= \frac{1}{t}\left(\int_t^{t+h} T(s)\,ds - \int_0^h T(s)\,ds\right)\\
&= \frac{1}{t}\left(\int_h^{t+h} T(s)\,ds - \int_0^t T(s)\,ds\right).
\end{aligned}$$

It follows that

$$\frac{T(t)-I}{t} = \frac{1}{t}\left(\int_h^{t+h} T(s)\,ds - \int_0^t T(s)\,ds\right)\left(\int_0^h T(s)\,ds\right)^{-1}$$

which converges to $A = (T(h)-I)\left(\int_0^h T(s)\,ds\right)^{-1}$ as $t \to 0$. We obtain that A is a bounded operator and that $D(A) = \overline{D(A)}$.

Exercise 4

We prove the assumption by using Proposition 3.2.8.

The linearity of A follows directly from the linearity of $A_k,\ k \in \{1,\dots,n\}$.

By assumption, there are $\omega_k \in \mathbb{R},\ \theta_k \in \left(\frac{\pi}{2},\pi\right),\ M_k > 0$, such that $\Sigma_{\omega_k,\theta_k} \subset \rho(A_k)$ and

$$\|R(\lambda, A_k)\|_{L(X_k)} \leq M_k|\lambda - \omega_k|^{-1}$$

for all $\lambda \in \Sigma_{\omega_k,\theta_k}$. Let $\left(x^{(k)}\right)_{k\in\mathbb{N}} = \left(\left(x_1^{(k)},\dots,x_n^{(k)}\right)\right)_{k\in\mathbb{N}} \subset D(A)$, $\lim\limits_{k\to\infty} x_k = x$ and $\lim\limits_{k\to\infty} Ax_k = y$. We want to show that $Ax = y$ (then, A is closed).

Note that $x_k \longrightarrow x$ implies that $x_i^{(k)} \longrightarrow x_i$ in $X_i,\ i \in \{1,\dots,n\}$, since

$$\|x_k - x\|_X^2 = \sum_{i=1}^n \left\|x_i^{(k)} - x_i\right\|_{X_i}^2 \longrightarrow 0.$$

Likewise, $Ax_k \longrightarrow y$ implies $A_i x_i^{(k)} \longrightarrow y_i$ in X_i.

We calculate

$$\begin{aligned}
\|Ax - y\|_X^2 &= \lim_{k\to\infty} \|A(x - x_k)\|^2\\
&= \lim_{k\to\infty} \sum_{i=1}^n \left\|A_i(x_i - x_i^{(k)})\right\|^2\\
&= \sum_{i=1}^n \lim_{k\to\infty} \left\|A_i(x_i - x_i^{(k)})\right\|^2\\
&\overset{A_i \text{ closed}}{=} \sum_{i=1}^n 0\\
&= 0,
\end{aligned}$$

hence $Ax = y$.

Let $\lambda \in \bigcap_{i=1}^{n} \rho(A_i)$ and

$$T\colon X \to D(A),\ x \mapsto \Big(R(\lambda, A_1)x_1, \dots, R(\lambda, A_n)x_n\Big).$$

We have

$$T(\lambda - A)x = T\Big((\lambda - A_1)x_1, \dots, (\lambda - A_n)x_n\Big) = (x_1, \dots, x_n) = x,$$

just as $(\lambda - A)Tx = x$. Thus, $T = R(\lambda, A)$.

Let $\omega := \max_{i \in \{1,\dots,n\}} \{\omega_i\} + 1$, then $\Pi := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \omega\} \subset \bigcap_{i=1}^{n} \rho(A_i)$.

Let $\lambda \in \Pi$ and $M := \max_{i \in \{1,\dots,n\}} \{M_i\}$. One has

$$\begin{aligned}
\|\lambda R(\lambda, A)x\|_X &= \sqrt{\sum_{i=1}^{n} \|\lambda R(\lambda, A_i)x_i\|^2} \\
&\overset{A_i \text{ sectorial}}{\leq} \sqrt{\sum_{i=1}^{n} \frac{|\lambda|^2}{|\lambda - \omega_i|^2} M_i^2 \|x_i\|^2} \\
&\leq M|\lambda| \sqrt{\sum_{i=1}^{n} \frac{1}{|\lambda - \omega|^2} \|x_i\|^2} \\
&= \frac{M|\lambda|}{|\lambda - \omega|} \|x\|,
\end{aligned}$$

thus $\|\lambda R(\lambda, A)\|_{L(X)} \leq \frac{M|\lambda|}{|\lambda - \omega|}$. As the latter is bounded for all $\lambda \in \Pi$, this ends the proof. □

Exercise 5

The original version of this exercise was incorrect. Most likely the exercise should have read as follows.

Exercise. *Let X be a real Banach space and $X_{\mathbb{C}}$ its complexification. Prove that in general $f\colon X_{\mathbb{C}} \to [0, \infty)$ given by $f(x + iy) = \sqrt{\|x\|^2 + \|y\|^2}$ does not satisfy the complex homogeneity property and therefore cannot be a norm on $X_{\mathbb{C}}$.*

Let $\lambda = \alpha + i\beta \in \mathbb{C}$ and $x + iy \in X_{\mathbb{C}}$. Then one has $\lambda(x + iy) = \alpha x - \beta y + i(\beta x + \alpha y)$. So if f were homogeneous, one would need to have

$$\begin{aligned}
\|\alpha x - \beta y\|^2 + \|\beta x + \alpha y\|^2 = f(\lambda(x + iy))^2 &= |\lambda|^2 f(x + iy)^2 \\
&= (\alpha^2 + \beta^2)(\|x\|^2 + \|y\|^2).
\end{aligned}$$

In particular, for $\alpha = \beta = 1$ this would mean

$$\|x - y\|^2 + \|x + y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all $x, y \in X$; in other words, X would satisfy the parallelogram law. Since only inner product spaces satisfy the parallelogram law (one can use the real polarisation identity to define an inner product that generates the norm in this case), the function f cannot be homogeneous in general for not all Banach spaces are Hilbert spaces.

Exercise 6

(i) A and $-A$ generate C_0-groups. Thus there exists $\omega \in \mathbb{R}$ such that $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subset \rho(\pm A - \omega)$ and $M > 0$ such that $\|R(\lambda, \pm A - \omega)\|_{L(X)} \leq \frac{M}{\operatorname{Re}\lambda}$. We use Proposition 3.2.8 to prove that A^2 is analytic. We can find $r_0 \geq 0$ and $\beta \in (0, \frac{\pi}{2})$ such that

$$\Pi := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > r_0\} \subset \{z^2 : z \in \Sigma_{0,\beta} + \omega\}.$$

Now take $\lambda \in \Pi$. Then there exists $\theta \in (-\beta, \beta)$ and $r \geq 0$ such that $\lambda = (re^{i\theta} + \omega)^2$. Now

$$\lambda I - A^2 = (re^{i\theta} + \omega - A)(re^{i\theta} + \omega + A)$$

which is bijective because $re^{i\theta} \in \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\} \subset \rho(\pm A - \omega)$. Also

$$\|\lambda R(\lambda, A^2)\| = \|\lambda R(re^{i\theta}, -A - w)R(re^{i\theta}, A - w)\| \leq |\lambda| \frac{M^2}{r^2(\cos\theta)^2} \leq \frac{|\lambda|}{r^2} \frac{M^2}{(\cos\beta)^2}.$$

where $\frac{|\lambda|}{r^2}$ is bounded.

(ii) We claim that A generates the C_0-group $\{T(t) : t \geq 0\}$, where for $f \in L^p(\mathbb{R})$ and $t \geq 0$ we define

$$T(t)f := f(\cdot + t). \tag{1}$$

Now for $f \in W^{1,p}(\mathbb{R})$ and $\varphi \in C_c^\infty(\mathbb{R})$ we have

$$\begin{aligned}
\int_{\mathbb{R}} f'(x)\varphi(x)\,dx &= -\int_{\mathbb{R}} f(x)\varphi'(x)\,dx \\
&= \lim_{h\to 0} \int_{\mathbb{R}} f(x)\left(\frac{\varphi(x-h) - \varphi(x)}{h}\right) dx \\
&= \lim_{h\to 0} \int_{\mathbb{R}} \left(\frac{f(x+h) - f(x)}{h}\right) \varphi(x)\,dx \\
&= \lim_{h\to 0} \int_{\mathbb{R}} \left(\frac{T(h)f(x) - f(x)}{h}\right) \varphi(x)\,dx
\end{aligned}$$

which implies that $\frac{T(h)f - f}{h}$ converges to f' in $L^p(\mathbb{R})$. On the other hand if we take $f \in L^p(\mathbb{R})$ such that $\frac{T(h)f-f}{h}$ converges in $L^p(\mathbb{R})$ to some function $u \in L^p(\mathbb{R})$, the above equalities also show that $f \in W^{1,p}(\mathbb{R})$ with $f' = u$.

(iii) It is $B = A^2$. Now use (i) and (ii).

Exercise 7

For each of the three subproblems we prove that the operator is sectorial. More precisely, we show in all three cases that the operator lies in $S(0, \theta_0, M)$ for all $\theta_0 \in (\pi/2, \pi)$ and a suitable $M = M(\theta_0) > 0$. Since in the lectures the resolvent set and resolvent were only introduced for closed operators, we would basically be required to establish closedness first, before we can address the sectoriality. However, our arguments yield the boundedness of the corresponding inverse operators which in particular implies the closedness of the operator.

In order to prove the claimed sectoriality, we have to find a solution in C^2_{b} of a non-homogeneous linear ODE and establish an estimate for the solution that allows us to deduce $\|R(\lambda, A)\| \leq \frac{M}{|\lambda|}$. We accomplish this by using the corresponding Green's functions. In the following we will omit a lot of straightforward calculation including the arguments to deduce the Green's functions. We refer to `http://math.stackexchange.com/q/442517` for a quick introduction on how to determine the Green's functions in this setting and to `http://www.greensfunction.unl.edu` for a useful directory with the respective formulas. Plenty of additional information can be found in the book *Theory of Ordinary Differential Equations* by Coddington and Levinson (1987) or for example in Chapter 3 in *Principles and techniques of applied mathematics* by Friedman (1962).

Let $\theta_0 \in (\pi/2, \pi)$ be fixed. Let $\lambda \in \Sigma_{0,\theta_0}$ and fix $\sqrt{\lambda} \in \mathbb{C}$ such that $\operatorname{Re}\sqrt{\lambda} > 0$. Note that there exists an $M \geq 1$ such that $|\operatorname{Im}\sqrt{\lambda}|^2 \leq (M-1)(\operatorname{Re}\sqrt{\lambda})^2$. Consequently $(\operatorname{Re}\sqrt{\lambda})^{-2} \leq M|\lambda|^{-1}$.

(a) Let $g \in C_{\mathrm{b}}(\mathbb{R})$. We claim that there exists a unique $f \in C^2_{\mathrm{b}}(\mathbb{R})$ such that $\lambda f - f'' = g$. By the basic theory for non-homogeneous linear ODE, the equation has a 2-dimensional solution space in $C^2(\mathbb{R})$, and any two such solutions differ by a linear combination of the functions $x \mapsto \exp(\sqrt{\lambda}x)$ and $x \mapsto \exp(-\sqrt{\lambda}x)$ (which form a basis of the solution space of the corresponding homogeneous equation). As any nontrivial linear combination of these two functions is unbounded, this shows that if there is a solution $f \in C^2_{\mathrm{b}}(\mathbb{R})$ it must be unique.
(Observe that for $\lambda \in (-\infty, 0]$, which is excluded here, the homogeneous equation has nontrivial solutions in $C^2_{\mathrm{b}}(\mathbb{R})$ and the non-homogeneous equation thus cannot be uniquely solvable in $C^2_{\mathrm{b}}(\mathbb{R})$.)
We next provide a formula for the unique solution $f \in C^2_{\mathrm{b}}(\mathbb{R})$. It is readily checked by a straightforward calculation and splitting the integral at x that

$$f(x) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\lambda}} \exp\bigl(-\sqrt{\lambda}|x-y|\bigr) g(y) \,\mathrm{d}y$$

is a solution in $C^2_{\mathrm{b}}(\mathbb{R})$. An elementary estimate shows

$$\|f\|_\infty \leq \|g\|_\infty \int_{-\infty}^{\infty} \frac{1}{2\operatorname{Re}\sqrt{\lambda}} \exp\bigl(-\operatorname{Re}\sqrt{\lambda}|y|\bigr) \,\mathrm{d}y = \frac{1}{(\operatorname{Re}\sqrt{\lambda})^2}\|g\|_\infty \leq \frac{M}{|\lambda|}\|g\|_\infty.$$

This establishes that the operator A is indeed sectorial and therefore generates an analytic semigroup.

(b) The functions in $C^2[0,1]$ given by $f_1(x) = \sinh(\sqrt{\lambda}x)$ and $f_2(x) = \sinh(\sqrt{\lambda}(1-x))$ are solutions of the homogeneous equation $\lambda f - f'' = 0$ with $f_1(0) = 0$ and $f_2(1) = 0$. The Wronskian $W(f_1, f_2) = f_1 f_2' - f_1' f_2$ is then easily seen to be constant on $[0,1]$ with value $c := c(\lambda) := f_1'(0) f_2(0) = -\sqrt{\lambda}\sinh(\sqrt{\lambda}) \neq 0$. Observe that $|c| \geq \operatorname{Re}\sqrt{\lambda}\sinh(\operatorname{Re}\sqrt{\lambda}) > 0$. Define $k \in C([0,1]\times[0,1])$ by

$$k(x,y) := \begin{cases} \frac{1}{c} f_1(x) f_2(y) & \text{for } 0 \leq x \leq y \leq 1, \\ \frac{1}{c} f_1(y) f_2(x) & \text{for } 0 \leq y \leq x \leq 1. \end{cases}$$

Let $g \in C[0,1]$. It can now readily be checked that f given by

$$f(x) := \int_0^1 k(x,y) g(y) \,\mathrm{d}y$$

is a solution of $\lambda f - f'' = g$ in $C^2[0,1]$ such that $f(0) = f(1) = 0$. As (f_1, f_2) is a basis of the solution space of the homogeneous equation (without boundary conditions), one easily deduces that f is the unique solution in $C^2[0,1]$ with $f(0) = f(1) = 0$ for the right hand side g.
To establish that the operator A is sectorial, it only remains to give a suitable estimate for the norm of f. Clearly

$$|f(x)| \leq \|g\|_\infty \int_0^1 |k(x,y)| \,\mathrm{d}y$$

for all $x \in [0,1]$. By elementary calculation and using that $|\sinh z| \le \cosh(\operatorname{Re} z)$, we obtain

$$
\begin{aligned}
\int_0^1 |k(x,y)|\,\mathrm{d}y &= \frac{1}{|c|}|f_2(x)| \int_0^x |f_1(y)|\,\mathrm{d}y + \frac{1}{|c|}|f_1(x)| \int_x^1 |f_2(y)|\,\mathrm{d}y \\
&\le \frac{1}{|c|}\Big(\cosh(\operatorname{Re}\sqrt{\lambda}(1-x)) \int_0^x \cosh(\operatorname{Re}\sqrt{\lambda}y)\,\mathrm{d}y + \cosh(\operatorname{Re}\sqrt{\lambda}x) \int_x^1 \cosh(\operatorname{Re}\sqrt{\lambda}(1-y))\,\mathrm{d}y\Big) \\
&\le \frac{\cosh(\operatorname{Re}\sqrt{\lambda}(1-x) \sinh(\operatorname{Re}\sqrt{\lambda}x) + \cosh(\operatorname{Re}\sqrt{\lambda}x)\sinh(\operatorname{Re}\sqrt{\lambda}(1-x))}{(\operatorname{Re}\sqrt{\lambda})^2 \sinh(\operatorname{Re}\sqrt{\lambda})} \\
&= \frac{1}{(\operatorname{Re}\sqrt{\lambda})^2} \le \frac{M}{|\lambda|}
\end{aligned}
$$

for all $x \in [0,1]$. This shows that A is indeed a sectorial operator in $C[0,1]$ and thus a generator of an analytic semigroup in $C[0,1]$.
The semigroup generated is not strongly continuous on $C[0,1]$ by Proposition 3.2.5 as $\overline{D(A)} \subset \{u \in C[0,1] : u(0) = u(1) = 0\} \ne C[0,1]$.

(c) The functions in $C^2[0,1]$ given by $f_1(x) = \cosh(\sqrt{\lambda}x)$ and $f_2(x) = \cosh(\sqrt{\lambda}(1-x))$ are solutions of the homogeneous equation $\lambda f - f'' = 0$ with $f_1'(0) = 0$ and $f_2'(1) = 0$. The Wronskian $W(f_1, f_2) = f_1 f_2' - f_1' f_2$ is then easily seen to be constant on $[0,1]$ with value $c := c(\lambda) := f_1(0)f_2'(0) = -\sqrt{\lambda}\sinh(\sqrt{\lambda}) \ne 0$. Observe that $|c| \ge \operatorname{Re}\sqrt{\lambda}\sinh(\operatorname{Re}\sqrt{\lambda}) > 0$. Define $k \in C([0,1]\times[0,1])$ by

$$
k(x,y) := \begin{cases} \frac{1}{c} f_1(x) f_2(y) & \text{for } 0 \le x \le y \le 1, \\ \frac{1}{c} f_1(y) f_2(x) & \text{for } 0 \le y \le x \le 1. \end{cases}
$$

Let $g \in C[0,1]$. It can now readily be checked that f given by

$$
f(x) := \int_0^1 k(x,y) g(y)\,\mathrm{d}y
$$

is a solution of $\lambda f - f'' = g$ in $C^2[0,1]$ such that $f'(0) = f'(1) = 0$. As (f_1, f_2) is a basis of the solution space of the homogeneous equation (without boundary conditions), one easily deduces that f is the unique solution in $C^2[0,1]$ with $f'(0) = f'(1) = 0$ for the right hand side g.
To establish that the operator A is sectorial, it only remains to give a suitable estimate for the norm of f. Clearly

$$
|f(x)| \le \|g\|_\infty \int_0^1 |k(x,y)|\,\mathrm{d}y
$$

for all $x \in [0,1]$. By elementary calculation we obtain

$$
\begin{aligned}
\int_0^1 |k(x,y)|\,\mathrm{d}y &= \frac{1}{|c|}|f_2(x)| \int_0^x |f_1(y)|\,\mathrm{d}y + \frac{1}{|c|}|f_1(x)| \int_x^1 |f_2(y)|\,\mathrm{d}y \\
&\le \frac{1}{|c|}\Big(\cosh(\operatorname{Re}\sqrt{\lambda}(1-x)) \int_0^x \cosh(\operatorname{Re}\sqrt{\lambda}y)\,\mathrm{d}y + \cosh(\operatorname{Re}\sqrt{\lambda}x) \int_x^1 \cosh(\operatorname{Re}\sqrt{\lambda}(1-y))\,\mathrm{d}y\Big) \\
&\le \frac{\cosh(\operatorname{Re}\sqrt{\lambda}(1-x) \sinh(\operatorname{Re}\sqrt{\lambda}x) + \cosh(\operatorname{Re}\sqrt{\lambda}x)\sinh(\operatorname{Re}\sqrt{\lambda}(1-x))}{(\operatorname{Re}\sqrt{\lambda})^2 \sinh(\operatorname{Re}\sqrt{\lambda})} \\
&= \frac{1}{(\operatorname{Re}\sqrt{\lambda})^2} \le \frac{M}{|\lambda|}
\end{aligned}
$$

for all $x \in [0,1]$. This shows that A is indeed a sectorial operator in $C[0,1]$ and thus a generator of an analytic semigroup on $C[0,1]$.
We claim that the semigroup generated by A on $C[0,1]$ is strongly continuous. By Proposition 3.2.5 it suffices to show that $\overline{D(A)} = C[0,1]$. As $D(A)$ is easily seen to be a unital $*$-subalgebra of $C[0,1]$ that separates points, it follows by Stone–Weierstraß that $D(A)$ is dense in $C[0,1]$. Hence the claim is established.