Solutions to the Exercises of Lecture 2

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Exercise 1.
Let X be a Banach space and $T \in \mathcal{L}(X)$ with $\|T\|_{L(X)} < 1$. Then the operator $I - T$ is invertible and its inverse is given by the Neumann series, i.e.,

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

Proof. First we check that the serie $\sum\limits_{k \in \mathbb{N}_0} T^k$ converges in $\mathcal{L}(X)$. Since $\|T\| < 1$, we have

$$\sum_{k=0}^{\infty} ||T^k|| \leq \sum_{k=0}^{\infty} ||T||^k = \frac{1}{1 - \|T\|} < \infty \ .$$

This implies that $\sum\limits_{k \in \mathbb{N}_0} T^k$ converges absolutely thus, it converges since $\mathcal{L}(X)$ is a Banach space. Now we prove that $\sum\limits_{k=0}^{\infty} T^k$ is the inverse of $I - T$. One has

$$\left(\sum_{k=0}^{n} T^k\right)(I-T) = \sum_{k=0}^{n} T^k - \sum_{k=0}^{n} T^{k+1} = \sum_{k=0}^{n} T^k - \sum_{k=1}^{n+1} T^k = T^0 - T^{n+1} = I - T^{n+1}, \tag{1}$$

similarely,

$$(I-T)\sum_{k=0}^{n} T^k = I - T^{n+1} \tag{2}$$

for each $n \in \mathbb{N}$. Since $\|T\| < 1$, it follows that $\lim\limits_{n\to\infty} T^n = 0$. Taking the limits for $n \to \infty$ in equations (1) and (2) we obtain

$$(I-T)\sum_{k=0}^{\infty} T^k = I = (\sum_{k=0}^{\infty} T^k)(I-T) \ .$$

Yields, $I - T$ is invertible and $(I-T)^{-1} = \sum\limits_{k=0}^{\infty} T^k$. □

Exercise 2.
On $X = \ell^p$ ($p \in [1, \infty)$) consider the operator A defined by $A(x_n) = (a_n x_n)$ for every $(x_n) \in D(A) = \{(x_n) \in \ell^p : (a_n x_n) \in \ell^p\}$, where $(a_n) \subset \mathbb{C}$ is a given sequence. Show that:

(i) A is bounded if and only if (a_n) is bounded.

(ii) A is a closed, densely defined operator.

(iii) A generates a C_0-semigroup $\{T(t)\}$ if and only if there exists $\omega \in \mathbb{R}$ such that $\operatorname{Re} a_n \leq \omega$ for all $n \in \mathbb{N}$. In this case, $T(t)(x_n) = (e^{ta_n} x_n)$ for each $(x_n) \in X$.

(iv) Prove that, if $a_n = -n^2$, then $\{T(t)\}$ is continuous in the operator norm on $(0, \infty)$ but not right continuous at $t = 0$.

Proof. Let $a = (a_n) \subset \mathbb{C}$ be the given sequence.

(i) If a is bounded, then

$$\|Ax\| = \|ax\| = \left(\sum_{n=0}^{\infty} |a_n x_n|^p\right)^{\frac{1}{p}} \leq \|a\|_{\ell^\infty} \|x\|_{ell^p}, \qquad \forall x \in \ell^p(\mathbb{N}).$$

Therefore, A is bounded and $\|A\| \leq \|a\|_{\ell^\infty}$. Conversely, assume that A is bounded. Consider, for $n \in \mathbb{N}$, $e^{(n)} \in \ell^p$ defined by $e^{(n)}_m = \delta_{nm}$ (Kronecker symbol). Then,

$$|a_n| = \|Ae^{(n)}\|_{\ell^p} \leq \|A\| \|e^{(n)}\|_{\ell^p} = \|A\|, \qquad \forall n \in \mathbb{N}.$$

Thus $a = (a_n)_n$ is bounded by $\|A\|$.

(ii) Let us show that $(A, D(A))$ is densely defined. Let $x = (x_n)_n \in \ell^p$ and define, for $n \in \mathbb{N}$, $S^{(n)}(x) \in \ell^p$ by

$$S^{(n)}_m(x) = \begin{cases} x_m, & m \leq n \\ 0, & m > n \end{cases}.$$

It is easy to see that $S^{(n)}(x) \in D(A)$, for all $n \in \mathbb{N}$, and $x - S^{(n)}(x) \underset{n\to\infty}{\longrightarrow} 0$ in ℓ^p.

For the closedness, let $(x^{(n)})_n$ be a sequence in $D(A)$, x and y belongs to ℓ^p such that $(x^{(n)}, Ax^{(n)}) \to (x, y)$ in $\ell^p \times \ell^p$. One has

$$|x^{(n)}_m - x_m| \leq \left(\sum_{k=1}^{\infty} |x^{(n)}_k - x_k|^p\right)^{1/p} = \|x^{(n)} - x\|_{\ell^p}, \qquad \forall m \in \mathbb{N}.$$

Hence $x^{(n)}_m \longrightarrow x_m$, as $n \to \infty$, for all $m \in \mathbb{N}$. Similarly, one obtains that $(Ax^{(n)})_m = a_m x^{(n)}_m \longrightarrow y_m$, as $n \to \infty$, for all $m \in \mathbb{N}$. Now, by the uniqueness of the limit one gets $y_m = a_m x_m$, for all $m \in \mathbb{N}$. Thus $y = Ax \in \ell^p$, i.e., $x \in D(A)$ and this ends the proof.

(iii) Next, we will show that $\sigma(A) = \overline{a(\mathbb{N})}$.

- $\sigma(A) \subset \overline{a(\mathbb{N})}$: Suppose $\lambda \notin \overline{a(\mathbb{N})}$. Then $a_\lambda := \lambda\not{1} - a$ is bounded away from zero and so its inverse is a bounded sequence. Denoting the multiplication operator corresponding to a sequence u by M_u, we have $M_{a_\lambda^{-1}} M_{a_\lambda} = M_{a_\lambda} M_{a_\lambda^{-1}} = I$, follows that $M_{a_\lambda} = \lambda I - A$ is invertible with inverse $M_{a_\lambda^{-1}}$.
- $\overline{a(\mathbb{N})} \subset \sigma(A)$: Take $\lambda = a_n \in a(\mathbb{N})$. Then $\lambda e_n - ae_n = 0$, so $\lambda = a_n$ is an eigenvalue of A. The claim follows because $\sigma(A)$ is closed, being the spectrum of a closed operator.

Now suppose that A generates a C_0-semigroup. Then $\sigma(A) = \overline{a(\mathbb{N})}$ is contained in the left halfplane $\{Re(\lambda < \omega)\}$, for some $\omega \in \mathbb{R}$. Hence $\mathrm{Re}(a_n) \leq \omega$, for all $n \in \mathbb{N}$. Conversely, suppose that $\mathrm{Re}(a_n) \leq \omega$ for some $\omega \in \mathbb{R}$ and all $n \in \mathbb{N}$. We show, in order to apply the Hille-Yosida theorem the resolvent estimate

$$\|R(\lambda, A)\| \leq \frac{1}{(\mathrm{Re}\,\lambda - \omega)}$$

for all $\mathrm{Re}\,\lambda > \omega$. But as we have seen above, $R(\lambda, A) = M_{a_\lambda^{-1}}$ for $a_\lambda = \lambda\not{1} - a$. Thus

$$\|R(\lambda, A)\| = \left\|M_{a_\lambda^{-1}}\right\| \leq \left\|a\lambda^{-1}\right\|_{\ell^\infty} \leq \frac{1}{(\mathrm{Re}\,\lambda - \omega)}.$$

The Hille-Yosida theorem shows that, in fact, A generates a stongly continuous semigroup in ℓ^p. For the explicite representation, consider the semigroup on $\ell^p(\mathbb{N})$ defined by

$$S(t)x = (e^{ta_n} x_n)_n, \qquad \forall x = (x_n)_n \in \ell^p.$$

Let us show first that $(S(t))_{t\geq 0}$ is strongly contiuous. Let $x = (x_n)_n \in \ell^p$. Then

$$\begin{cases} \lim_{t\to 0} e^{ta_n} x_n = x_n, & \forall n \in \mathbb{N} \\ |e^{ta_n} x_n| \leq e^{\mathrm{Re}\, a_n} |x_n| \leq e^{\omega t} |x_n| \end{cases}.$$

By dominated convergence theorem $S(t)x \longrightarrow x$ in ℓ^p. Let $(B, D(B))$ be the generator of $(S(t))_{t\geq 0}$. In order to prouve that $A \equiv B$, it is enough to show that $D(A) \subset D(B)$ and $B_{|D(A)} = A$, since both A and B are generators of semigroups. To do so, let $x = (x_n)_n \in D(A)$, i.e. $(a_n x_n)_n \in \ell^p$. Then, by the mean value Theorem,

$$\begin{cases} \lim_{t\to 0} \dfrac{e^{ta_n} x_n - x_n}{t} = a_n x_n, & \forall n \in \mathbb{N} \\ \left| \dfrac{e^{ta_n} x_n - x_n}{t} \right| \leq M a_n x_n, & \forall n \in \mathbb{N},\ \forall t \in (0,1] \end{cases},$$

where $M = \sup_{t\in(0,1)} |e^{ta_n}|$. Hence, by the convergence dominated theorem $\dfrac{S(t)x - x}{t} \longrightarrow Ax$ in ℓ^p. Thus $x \in D(B)$ and $Bx = Ax$.

(iv) Let $t, \varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $e^{-tn^2} < \varepsilon$, hence $\|T(t)e^{(n)}\| = e^{-tn^2} < \varepsilon$ where $e^{(n)}$ is the sequence defined in (i). This shows that $T(t)$ does not converge uniformly to $T(0) = I$ as $t \to 0$.

Now consider the sequence $u_k := e^{-ptk^2}(1 - e^{-hk^2})$, $k \in \mathbb{N}$, where $h > 0$. Let $\varepsilon > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that $u_k < \varepsilon$ for all $k \geq k_0$, independent of h. (The independence of h can be achieved because $1 - e^{-hk^2}$ is bounded by 2). So for $x \in \ell^p(\mathbb{N})$ with $\|x\| \leq 1$ we have

$$\begin{aligned} \|T(t)x - T(t+h)x\|^p &= \sum_{k=1}^{\infty} e^{-tpk^2}(1 - e^{-hk^2})^p |x_k|^p \\ &\leq \sum_{k=1}^{k_0} e^{-tpk^2}(1 - e^{-hk^2})^p |x_k|^p + \sum_{k=k_0+1}^{\infty} \varepsilon |x_k|^p. \\ &\leq \sum_{k=1}^{k_0} e^{-tpk^2}(1 - e^{-hk^2})^p + \varepsilon. \end{aligned}$$

Since the first term tends to zero for $h \to 0$, this shows uniform continuity of $T(\cdot)$ away from zero.

□

Exercise 3.
Let $X = C_0(\mathbb{R})$ and $q \in C(\mathbb{R})$. Consider the operator $(Af)(s) := q(s)f(s)$ with $D(A) := \{f \in X : qf \in X\}$ and make analogous statements as in the previous exercise. Prove these statements.

Solution:
The analogous statements to the previous exercise are:

(i) A is bounded if and only if q is bounded.

(ii) A is a closed operator with dense domain.

(iii) A generates a C_0-semigroup if and only if there exists $\omega \in \mathbb{R}$ such that $\mathrm{Re}\, q \leq \omega$. In this case, $T(t)f = e^{tq}f$ for each $f \in C_0(\mathbb{R})$.

(iv) If $q(x) = -x^2$, then $\{T(t)\}$ is continuous in the operator norm on $(0,\infty)$ but not right continuous at $t = 0$.

Proof. (i) If q is bounded, $\|Af\|_\infty = \|qf\|_\infty \le \|q\|_\infty \|f\|_\infty$ for all $f \in \mathrm{C}_0(\mathbb{R})$. Hence, A is bounded and $\|A\| \le \|q\|_\infty$. Conversely, assume that A is bounded, Let $\phi_n \subset C_0(\mathbb{R})$ given by

$$\phi_n(x) = \begin{cases} 1, & if \quad x \in [-n, n] \\ -\frac{1}{n}x + 2, & if \quad |x| \in [n, 2n] \\ 0 & if \quad |x| \ge 2n \end{cases}.$$

Then

$$\sup_{x \in [-n,n]} |q(x)| \le \|q\phi_n\|_\infty = \|A\phi_n\|_\infty \le \|A\| \|\phi_n\|_\infty = \|A\|,$$

for all $n \in \mathbb{N}$. Therefore, q is bounded and $\|q\|_\infty \le \|A\|$.

(ii) A is densely defined since the space of continuous function with compact support $\mathrm{C_c}(\mathbb{R})$ is contained in $D(A)$ and it is dense in $C_0(\mathbb{R})$. Let us show that $(A, D(A))$ is closed. Let $(f_n)_n \subset D(A)$, $f, g \in C_0(\mathbb{R})$ such that $(f_n, Af_n) \to (f, g)$ in $C_0(\mathbb{R})$. Then, for every $x \in \mathbb{R}$, $f_n(x) \longrightarrow f(x)$ and $q(x)f(x) \longrightarrow g(x)$. The uniqueness of the limit in $\mathbb{R}$ implies that $g = qf \in C_0(\mathbb{R})$, hence $f \in D(A)$ and $g = Af$.

(iii) Next, we will show that $\sigma(A) = \overline{q(\mathbb{R})}$.

- $\sigma(A) \subset \overline{q(\mathbb{R})}$: Suppose $\lambda \notin \overline{q(\mathbb{R})}$. Then $p := \lambda \nVdash - q$ is bounded away from zero thus, its inverse is bounded. Denoting the multiplication operator corresponding to a function u by M_u, we have $M_{p^{-1}} M_p = M_p M_{p^{-1}} = I$ on $D(A)$, so $M_p = \lambda I - A$ is invertible with inverse $M_{p^{-1}}$.
- $\overline{q(\mathbb{R})} \subset \sigma(A)$: Take $\lambda = q(x) \in q(\mathbb{R})$. Then $p := \lambda \nVdash - q$ vanishes in x. So if we choose functions f_n, $n \in \mathbb{N}$ such that $f_n(x) = 1$ and $\mathrm{supp}(f_n) \subset B_{1/n}(x)$, then $\lambda f_n - Af_n = pf_n \to 0$, as $n \to \infty$. However, $\|f_n\| = 1$. This shows λ is in the approximate point spectrum of A. Having now shown that $q(\mathbb{R}) \subset \sigma(A)$, the claim follows since $\sigma(A)$ is closed.

Now suppose A generates a C_0-semigroup. Then $\sigma(A) = \overline{q(\mathbb{R})}$ is contained in the left halfplane $\{\mathrm{Re}(\lambda) \le \omega\}$ for some $\omega \in \mathbb{R}$. This shows that $\mathrm{Re}(q(x)) \le \omega$, for all $x \in \mathbb{R}$.

Conversely, suppose that $\mathrm{Re}(q(x)) \le \omega$ for some $\omega \in \mathbb{R}$ and all $x \in \mathbb{R}$. We show, in order to apply the Hille-Yosida theorem, the resolvent estimate

$$\|R(\lambda, A)\| \le \frac{1}{\mathrm{Re}\,\lambda - \omega}$$

for all λ such that $\mathrm{Re}(\lambda) > \omega$. But as we have seen above, $R(\lambda, A) = M_{p^{-1}}$ for $p = \lambda \mathbb{1} - q$, hence

$$\|R(\lambda, A)\| = \left\|M_{p^{-1}}\right\| = \left\|M_{p^{-1}}\right\| = \left\|p^{-1}\right\| \le \frac{1}{\mathrm{Re}\,\lambda - \omega}$$

The Hille-Yosida theorem shows that A generates a strongly continuous semigroup. For the explicite representation, consider the semigroup on $C_0(\mathbb{R})$ defined by

$$S(t)f = e^{tq}f, \qquad f \in C_0(\mathbb{R})$$

Let us show that $(S(t))_{t \ge 0}$ is strongly continuous. To do so, let us show that $e^{tq}f$ converges uniformly to f for $f \in C_c(\mathbb{R})$ of compact support. Let $f \in C_c(\mathbb{R})$ such that $\mathrm{supp}(f) \subseteq [a, b]$. Then,

$$\begin{aligned} |e^{tq(x)}f(x) - f(x)| &\le |e^{tq(x)} - 1||f(x)| \\ &\le q(x)t \sup_{s \in [0,1]} (e^{sq(x)})|f(x)| \\ &\le Mt\|f\|_\infty, \end{aligned}$$

where $M = \sup_{[a,b]}(q)\sup\{e^{sq(x)} : (x,t) \in [a,b]\times[0,1]\} < \infty$. Therefore, $\|e^{tq}f - f\|_\infty \longrightarrow 0$ as $t \to 0$. For the general setting, we conclude by the density of $C_c(\mathbb{R})$ in $C_0(\mathbb{R})$. Let us identify the generator $(B, D(B))$ of $(S(t))_{t\geq 0}$. Let $f \in D(B)$, one has

$$\lim_{h\to 0}\frac{e^{tq}f - f}{h} = Bf.$$

Since the uniform convergence implies the pointwise one then,

$$\lim_{h\to 0}\frac{e^{tq}f - f}{h}(x) = q(x)f(x)$$

for all $x \in \mathbb{R}$. It follows that $Bf = qf$ and thus $D(B) \subset D(A)$ and $Af = Bf$ for all $f \in D(B)$. The equality $D(A) = D(B)$ follows as in Exercise 2.

(iv) Let $t, \varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $e^{-tx^2} < \varepsilon$ for $|x| > n$. Let f be a continuous function such that $\|f\| = 1$ and $\mathrm{supp}(f) \subset \{x \in \mathbb{R} \mid |x| > n\}$. Then $\|T(t)f\| \leq \varepsilon$, showing that $T(t)$ does not converge uniformly to $T(0) = I$ for $t \to 0$.

Now consider the function $g(x) = e^{-tx^2}(1 - e^{-hx^2})$ for $h > 0$. For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $|g(x)| < \varepsilon$ for $|x| > n$, (n can be chosen to be independent of h because $1 - e^{-hx^2}$ is bounded by 2). So for $f \in C_0(\mathbb{R})$ with $\|f\| \leq 1$ we have

$$\|T(t)f - T(t+h)f\| = \sup_{x\in\mathbb{R}} g(x)|f(x)| \leq \max\left\{\sup_{x\in[-n,n]} g(x), \varepsilon\right\} \leq \max\left\{1 - e^{-hn^2}, \varepsilon\right\}.$$

For $h \to 0$, this tends to 0, showing uniform continuity of $T(\cdot)$ away from zero.

□

Exercise 4.
On the Banach space $X_0 := \{f \in \mathrm{C}([0,1]) : f(1) = 0\}$ endowed with the sup-norm, consider the family $\{T(t)\}$ of operators, defined by

$$(T(t)f)(x) := \begin{cases} f(x+t), & \text{if } x+t \leq 1,\\ 0, & \text{otherwise,}\end{cases}$$

for each $x \in [0,1]$ and $t \geq 0$. Prove that $\{T(t)\}$ is a C_0-semigroup, called the semigroup of left-translations on $\mathrm{C}([0,1])$, and show that its growth bound ω_0 is $-\infty$.

Proof. We first check the semigroup property. Clearly $T(0)$ is the identity operator. For $t, s \geq 0$, $x \in [0,1]$ and $f \in X_0$ we obtain

$$(T(t)T(s)f)(x) = \begin{cases} (T(s)f)(x+t), & \text{if } x+t \leq 1,\\ 0, & \text{if } x+t > 1.\end{cases}$$

and

$$(T(s)f)(x+t) = \begin{cases} f(x+t+s), & \text{if } x+t+s \leq 1,\\ 0, & \text{if } x+t+s > 1.\end{cases}$$

This implies

$$\begin{aligned}(T(t)T(s))f(x) &= \begin{cases} f(x+t+s), & \text{if } x+t+s \leq 1,\\ 0, & \text{if } x+t+s > 1.\end{cases}\\ &= (T(t+s)f)(x)\end{aligned}$$

for all $t, s \geq 0$, $x \in [0,1]$ and $f \in X_0$.

We now prove that $\{T(t)\}$ is strongly continuous. By the semigroup property it suffices to check that

$$\lim_{t\to 0^+} \|T(t)f - f\| = 0$$

for each $f \in X_0$ (see Proposition I.5.3 in [EN] for a proof of this fact). Let $f \in X_0$ and take $\varepsilon > 0$. By the uniform continuity of f we find $\delta > 0$ such that

$$|f(x) - f(y)| \le \varepsilon$$

for all $x, y \in [0,1]$ with $|x - y| < \delta$. Since f vanishes at $x = 1$ we may assume (by making δ smaller if necessary) that $|f(x)| \le \varepsilon$ for all $x \in [0,1]$ with $x > 1 - \delta$. We then obtain

$$\begin{aligned}\|T(t)f - f\| &= \sup_{x\in[0,1]} |(T(t)f)(x) - f(x)| = \max\left(\sup_{x\in[0,1-t]} |f(x+t) - f(x)|, \sup_{x\in(1-t,1]} |0 - f(x)|\right)\\ &= \max\left(\sup_{x\in[0,1-t]} |f(x+t) - f(x)|, \sup_{x\in(1-t,1]} |f(x)|\right) \le \varepsilon\end{aligned}$$

for all $t \in [0, \delta)$.

We clearly have $\|T(t)\| = \|0\| = 0$ for each $t > 1$. Thus, given any $\omega \in \mathbb{R}$ we have $||T(t)|| \le Me^{\omega t}$ for all $t \ge 0$ if we set

$$M := \max\left(1, \max_{t\in[0,1]} e^{-\omega t}||T(t)||\right) \ge 1.$$

As a consequence we obtain that the growth bound ω_0 of $\{T(t)\}$ is equal to $-\infty$. Note that this is true for any nilpotent C_0-semigroup, i.e., each C_0-semigroup $\{T(t)\}$ with $T(t) = 0$ for some $t > 0$. □

In the solution of the next exercise we use the following version of the Picard-Lindelöf theorem.

Theorem 1. *Let $F : [0,1] \times \mathbb{C} \to \mathbb{C}$ be a continuous map that is Lipschitz-continuous in the second component, i.e. we have a constant $M \ge 0$ such that*

$$|F(t, y_1 - y_2)| \le M|y_1 - y_2|$$

for all $t \in [0,1]$ and $y_1, y_2 \in \mathbb{C}$. Then for each $x_0 \in \mathbb{C}$ there exists a unique solution $y : [0,1] \to \mathbb{C}$ for the initial value problem

$$y' = F(\,\cdot\,, y), \quad y(0) = x_0.$$

Exercise 5.
Let $X := C([0,1])$. Prove that

(a) the spectrum of the operator $A_1 : \{y \in C^1([0,1]) \mid y(0) = 0\} \subseteq X \to X$, defined by $A_1 y := y'$ for $f \in D(A_1)$, is empty.

(b) the spectrum of the operator $A_2 : \{y \in C^1([0,1]) \mid y(0) = y(1) = 0\} \subseteq X \to X$, defined by $A_1 y := y'$ for $f \in D(A_1)$, equals $\mathbb{C}$.

Proof. To show (a), we prove that the operator $(\lambda - A_1)$ is bijective and $(\lambda - A_1)^{-1} \in \mathcal{L}(X)$ for arbitrary $\lambda \in \mathbb{C}$. Applying Theorem 1 shows that for a given $f \in C([0,1])$ and $\lambda \in \mathbb{C}$ the inital value problem

$$y' = \lambda y - f, \quad y(0) = 0$$

has a unique solution $y\colon [0,1] \to \mathbb{C}$. In other words: For each $\lambda \in \mathbb{C}$ and $f \in C([0,1])$ there exsits a unique $y \in D(A_1)$ such that $(\lambda - A_1)y = f$. Thus $(\lambda - A_1)$ is bijective for each $\lambda \in \mathbb{C}$. To prouve that $(\lambda - A_1)^{-1} \in \mathcal{L}(X)$, thanks to the graph Theorem, it suffices to prouve the closedness of $(A_1, D(A_1))$. Let $(f_n)_n \subset D(A_1)$, $(f, g) \in X$ such that $f_n \to f$ and $Af_n \to g$ in X. Thus $(f_n^{'})_n$ converges uniformly in $[0,1]$ to g. Therefore

$$f_n(x) = \int_0^x f_n^{'}(t)dt \Longrightarrow f(x) = \lim_{n\to\infty} f_n(x) = \int_0^x g(t)dt.$$

Thus $f \in C^1([0,1])$, $f(0) = 0$ and $f^{'}(x) = g(x)$, i.e., $f \in D(A_1)$ and $A_1 f = g$ which shows the closedness.

To show (b), we prove that the operator $(\lambda - A_2)$ is not surjective for any $\lambda \in \mathbb{C}$. For $\lambda = 0$ it is trivial that $\lambda - A_2$ is not surjective (choose f continue but not of classe C^1). For $\lambda \in \mathbb{C}\backslash\{0\}$ we now have to find a function $g \in C([0,1])$ such that there does not exist $y \in D(A_2)$ satisfying $(\lambda - A_2)y = g$. Set $g(t) := t$ and observe that the initial value problem

$$(\lambda - A_2)y(t) = g(t), \quad y(0) = 0 \;\Leftrightarrow\; y'(t) = \lambda y(t) + t, \quad y(0) = 0, \qquad \forall t \in [0,1]$$

has the unique solution $y(t) = e^{\lambda t}(-\lambda^{-1}e^{-\lambda t}t - \lambda^{-2}e^{-\lambda t} + \frac{1}{\lambda^2})$. However,

$$y(1) = -\lambda^{-1} - \lambda^{-2} + \tfrac{e^\lambda}{\lambda^2} = \frac{e^\lambda - \lambda - 1}{\lambda^2} \neq 0 \quad \text{for} \quad \lambda \neq 0,$$

i.e. $y \notin D(A_2)$ and thus $\lambda - A_2$ is not invertible for any $\lambda \in \mathbb{C}$. □

Exercise 6.
Determine which of the following operators generate a C_0-semigroup on $X = C([0,1])$.

(i) $A_1 f := f'$ for $f \in D(A_1) := \{f \in C^1([0,1]) : f(0) = 0\}$.

(ii) $A_2 f := f''$ for $f \in D(A_2) := C^2([0,1])$.

(iii) $A_3 f := f''$ for $f \in D(A_3) := \{f \in C^2([0,1]) : f(0) = f(1) = 0\}$.

(iv) $A_4 f := f''$ for $f \in D(A_4) := \{f \in C^2([0,1]) : f''(0) = 0\}$.

Solution. Neither of the operators $(A_1, D(A_1))$ and $(A_3, D(A_3))$ is densely defined and thus they cannot generate C_0-semigroups. To see this, consider any continuous function $f \in C([0,1])$ with $f(0) = 1$. Then

$$||f - g|| = \sup_{x\in[0,1]} |f(x) - g(x)| \geq |f(0) - g(0)| = 1$$

for all $g \in D(A_1) \supseteq D(A_3)$.
Both of the remaining operators have empty resolvent set and thus cannot generate C_0-semigroups either. To see this, let $\lambda \in \mathbb{C}$ and consider the operator $\lambda - A_i$ for $i \in \{2,4\}$. Consider $\mu \in \mathbb{C}$ such that $\mu^2 = \lambda$ and define $f(x) := \sinh(\mu x)$ for $x \in [0,1]$. Then $f \in C^2([0,1])$ and $f''(0) = \lambda f(0) = 0$, i.e., $f \in D(A_4) \subset D(A_2)$. Moreover, f solves the equation $f'' = \lambda f$, i.e.,

$$(\lambda - A_i)f = 0$$

for $i = 1,2$. Hence $(\lambda - A_i)$ is not injective and we conclude $\sigma(A) = \mathbb{C}$. □

Exercise 7. On $X := C_0(\mathbb{R})$ consider the one-parameter semigroup $\{T(t)\}$ defined by

$$(T(t)f)(x) = \exp\left(\int\limits_{x-t}^{x} q(s)\,ds\right) f(x-t)$$

for $x \in \mathbb{R}$, $t \geq 0$, where $q\colon \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function. Prove that $\{T(t)\}$ is a C_0-semigroup on X and identify its infinitesimal generator.

Proof. We first check the semigroup property. For $t, r \geq 0$, $x \in \mathbb{R}$ and $f \in X$ we compute

$$(T(0)f)(x) = \exp\left(\int_x^x q(s)\,ds\right) f(x) = \exp(0)f(x) = f(x)\ ,$$

and

$$\begin{aligned}
(T(t)T(r)f)\,(x) &= T(t)\left(x \mapsto \exp\left(\int_{x-r}^{x} q(s)\,ds\right) f(x-r)\right)(x) \\
&= \exp\left(\int_{x-t}^{x} q(s)\,ds\right) \exp\left(\int_{x-r-t}^{x-t} q(s)\,ds\right) f(x-r-t) \\
&= \exp\left(\int_{x-t}^{x} q(s)\,ds \ + \int_{x-r-t}^{x-t} q(s)\,ds\right) f(x-r-t) \\
&= \exp\left(\int_{x-r-t}^{x} q(s)\,ds\right) f(x-r-t) = (T(t+r)f)\,(x)\ .
\end{aligned}$$

We now prove that $\{T(t)\}$ is strongly continuous. By the semigroup property it suffices to check that

$$\lim_{t\to 0^+} \|T(t)f - f\| = 0$$

for each $f \in X$ (see Proposition I.5.3 in [EN] for a proof of this fact). Let $f \in X$ and $\varepsilon > 0$. By the uniform continuity of f there exists $0 < \delta < 1$ such that

$$|f(x-t) - f(x)| < \varepsilon/2, \qquad \forall t \in [0, \delta).$$

On the other hand, since q is bounded then

$$-t\|q\|_\infty \leq \int_{x-t}^{x} q(s)ds \leq t\|q\|_\infty, \qquad \forall (t,x) \in [0,\delta) \times \mathbb{R}.$$

Thus,

$$e^{-t\|q\|_\infty} - 1 \leq \exp\left(\int_{x-t}^{x} q(s)ds\right) - 1 \leq e^{t\|q\|_\infty} - 1, \qquad \forall (t,x) \in [0,\delta) \times \mathbb{R}.$$

Since $\lim\limits_{t\to 0}(e^{-t\|q\|_\infty} - 1) = \lim\limits_{t\to 0}(e^{t\|q\|_\infty} - 1) = 0$. Then

$$\sup_{x\in\mathbb{R}} \left|\exp\left(\int_{x-t}^{x} q(s)ds\right) - 1\right| \leq \varepsilon/2.$$

Therefore we have by the triangular inequality

$$\begin{aligned}
\|T(t)f - f\| &= \sup_{x\in\mathbb{R}} |(T(t)f)(x) - f(x)| \\
&= \sup_{x\in\mathbb{R}} \left|\exp\left(\int_{x-t}^{x} q(s)\,ds\right) f(x-t) - f(x)\right| \\
&= \sup_{x\in\mathbb{R}} \left|\exp\left(\int_{x-t}^{x} q(s)\,ds\right) f(x-t) - f(x-t) + f(x-t) - f(x)\right| \\
&\leq \sup_{x\in\mathbb{R}} \left|\exp\left(\int_{x-t}^{x} q(s)\,ds\right) - 1\right| |f(x-t)| + \sup_{x\in\mathbb{R}} |f(x-t) - f(x)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon\ .
\end{aligned}$$

Now, let us prouve that

$$Af := q \cdot f - f' \quad \text{with domain} \qquad D(A) := C_0^1(\mathbb{R})$$

is the generator of $\{T(t)\}$. Since minus the first derivate on $C_0^1(\mathbb{R})$ is the generator of the semigroup left translation in $C_0(\mathbb{R})$ with domain $C_0^1(\mathbb{R})$ and q is bounded, it follows that operator A is a bounded perturbation of a generator of semigroup, hence by [EN, Theorem III.1.3], A is a generator of a strongly continuous semigroup. Let us consider $(B, D(B))$ the generator of $\{T(t)\}$. We have by the product rule, for $f \in D(A)$,

$$\begin{aligned}
\frac{d}{dt}T(t)f(x)|_{t=0} &= \frac{d}{dt}\exp\left(\int_{x-t}^{x} q(s)\,ds\right) f(x-t)|_{t=0} \\
&= \exp\left(\int_{x}^{x} q(s)\,ds\right) q(x)f(x) - f'(x) \\
&= q(x)f(x) - f'(x)
\end{aligned}$$

for all $x \in \mathbb{R}$. Yields $Af = Bf$ for all $f \in D(A)$. Thus $D(A) \subseteq D(B)$. The equality $D(A) = D(B)$ follows immediately since as in Exercise 2. Hence A is generator of $\{T(t)\}$. □

Exercise 8.
Let $A : D(A) \subseteq X \to X$ be a closed, densely defined linear operator on a Banach space X. Then the following properties are equivalent:

(a) A generates a C_0-group $(T(t))_{t\in\mathbb{R}}$.

(b) For some $w \geq 0$ we have

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > \omega\} \cup \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) < -\omega\} \subseteq \mathbb{C}\backslash\sigma(A)$$

and for some $M \geq 0$

$$\|R(\lambda, A)^n\| \leq M\tfrac{1}{(|\operatorname{Re}(\lambda)|-\omega)^n} \quad \forall |\operatorname{Re}(\lambda)| > \omega, n \in \mathbb{N}.$$

Proof. The implication $(a) \Rightarrow (b)$ follows applying Theorem 2.1.15 (Hille-Yosida) to A and $-A$. For $(b) \Rightarrow (a)$ we conclude by Theorem 2.1.15 that both A and $-A$ are generators of two strongly continuous semigroups $(T_+(t))_{t\geq 0}, (T_-(t))_{t\geq 0}$.
Furthermore, the Yosida approximation of A, denoted by $A_n^+ := nAR(n, A)$, and the Yosida approximation of $-A$, denoted by $A_n^- := n(-A)R(n, -A)$ commute. As seen in the proof of Theorem 2.1.15 we have

$$T_+(t)x = \|\cdot\|_X - \lim_{n\to\infty} \sum_{k=0}^{\infty} \tfrac{t^k}{k!} A_n^{+k} x$$

for all $x \in X$, and the same holds for $(T_-(t))_{t\geq 0}$. Thus, we see that $(T_+(t))_{t\geq 0}$ and $(T_-(t))_{t\geq 0}$ commute. Moreover the product of the two commuting semigroups yields another semigroup

$$U(t) := T_+(t)T_-(t), \qquad \forall t \geq 0,$$

and has a generator B satisfying

$$Bx = Ax + (-A)x = 0$$

for all $x \in D(A) \cap D(-A) = D(A) \subseteq D(B)$. Thus for every $x \in D(A)$

$$U(t)x - x = B\int_0^t U(s)x ds = 0 \Longrightarrow U(t)x = x. \tag{3}$$

By the density of $D(A)$, (3) holds for all $x \in X$. In particular, we have $T_-(t) = T_+(t)^{-1}$ for all $t \geq 0$. Hence,

$$T(t) := \begin{cases} T_+(t) & \text{for } t \geq 0 \\ T_-(-t) & \text{for } t < 0 \end{cases}$$

forms a C_0-group which is strongly continuous whose generator is A. □

References

[EN] Engel, K., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Springer 1999.