Degenerate Self-adjoint Evolution Equations

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The study of degenerate evolution equations started in the fifties of the last century and has been the object of many researches in a wide generality, since the seminal works of W. Feller in one dimension and J.J. Kohn and L. Nirenberg in higher dimensions. Of particular interest is the case of degeneracy for parabolic equation, and the results heavily depend on the kind of degeneracy. The study of degenerate parabolic equation has been the object of intensive research during the last few decades. In 1998 Campiti et all [1] established some qualitative properties of the degenerate parabolic operators which are the most relevant results that were available in the literature at that time. Precisely, they considered the following Problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{\partial}{\partial x}\big(a(x)\frac{\partial u}{\partial x}\big)(t,x) - b(x)u(t,x), & (t,x) \in (0,+\infty)\times(0,1), \\ u(0,x) = u_0(x), & x \in (0,1), \\ \lim\limits_{x\longrightarrow 0,1} a(x)\dfrac{\partial u}{\partial x}(t,x) = 0, & t \in (0,+\infty). \end{cases} \tag{1}$$

Here $a \in \mathcal{C}^1([0;1])$ is a non negative function vanishing only at $0,1$ and $b \in \mathcal{C}([0;1])$.

In this project we are interested to the existence, qualitative properties, asymptotic behaviour and approximation of the solution of Problem (1). More precisely, by using abstract semigroups theory we show that the operator $\mathcal{M}$ defined by

$$\mathcal{M}u = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) - bu, \tag{2}$$

with a suitable domain, generates a C_0-semigroup $(T(t))_{t\geq 0}$ in $L^p(0,1), 1 \leq p < \infty$ and in $\mathcal{C}([0;1])$. Under further hypotheses on the function a, analyticity, compactness and differentiability of this semigroup will also be studied.

In the second step we discuss some qualitative properties of the generator $\mathcal{M}$, namely, proving a characterization of its invertibility and a condition for the summability of the eigenvalues which in turn gives an interesting trace formula if $b = 0$.

In the following step, we study the asymptotic behaviour of the semigroup $(T(t))_{t\geq 0}$ if $b \geq 0$. In the particular case $b = 0$, we prove that the semigroup converges strongly to the projection $Pf = \int_0^1 f$ in $L^p(0;1)$, $1 \leq p < \infty$. By means of suitable integrability properties of the function $\dfrac{1}{a}$ we characterize the norm convergence of this semigroup.

Finally, we consider the problem of approximating the solution of Problem (1) for $b = 0$. We introduce a suitable sequence of convolution operators and we show the connexion with the semigroup $(T(t))_{t\geq 0}$ using Trotter's theorem for which details can be found in the reference [2].

References

[1] M. Campiti, G. Metafune, and D. Pallara, Degenerate self-adjoint evolution equations on the unit interval, Semigroup Forum, **57** (1998), 1-36.

[2] K. J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations Springer Verlag, Berlin (2000).