A CLASS OF UNIFORMLY ELLIPTIC LINEAR OPERATORS WITH HOMOGENEOUS NEUMANN BOUNDARY CONDITION

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In the present project we consider a class of uniformly elliptic linear operators

$$\mathcal{A} = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i=1}^{N} F_i D_i - V,$$

with smooth and possibly unbounded coefficients in smooth convex open sets $\Omega \subset \mathbb{R}^N$. Under suitable assumptions on q_{ij}, F_i and V, we prove that the Cauchy problem associated with the operator $\mathcal{A}$ with homogeneous Neumann boundary condition admits a unique bounded classical solution u for any initial datum f which is bounded and continuous in $\overline{\Omega}$. We also prove uniform and pointwise gradient estimates for u. To do this, our main assumptions are a dissipativity condition on the drift F, a Lyapunov type condition ensuring that a maximum principle holds, and that V is bounded from below. Concerning the existence, we consider the solutions of Neumann problems in a nested sequence Ω_n of bounded domains whose union is Ω, and show that u_n converge to the desired solution u. Setting $P_t f(x) = u(t,x)$, it turns out that (P_t) is a semigroup in the space of bounded and continuous functions in $\overline{\Omega}$. By using the Bernstein method, we prove the following uniform gradient estimates for P_t

$$\|\nabla P_t f\|_\infty \le \frac{C_T}{\sqrt{t}} \|f\|_\infty, \quad 0 < t \le T, \; f \in C_b(\overline{\Omega})$$

$$\|\nabla P_t f\|_\infty \le C_T\big(\|f\|_\infty + \|\nabla f\|_\infty\big), \quad 0 < t \le T, \; f \in C_b^1(\overline{\Omega}), \text{with } \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

As a consequence, the domain of the weak generator of (P_t) is contained in $C_b^1(\overline{\Omega})$. Assuming some restrictions on the coefficients of $\mathcal{A}$, we prove the pointwise estimates

$$|\nabla P_t f(x)|^p \le e^{ktp} P_t(|\nabla f|^p)(x)$$

which are of much help in the study of (P_t) in the spaces $L^p(\Omega, \mu)$, $1 \le p < \infty$, in the case where there exists an invariant measure μ for (P_t). The above results are contained in the paper [1]. The case $\Omega = \mathbb{R}^N$ has been treated in [3] by the same method and in [2] by probabilistic tools.

References

[1] M. Bertoldi, S. Fornaro: *Gradient estimates in parabolic problems with unbounded coefficients*, Studia Math. **165** (2004) 221–254.

[2] S. Cerrai: *Second order PDS's in finite and infinite dimension*,Lecture Notes in Math. 1762, Springer, Berlin, 2001.

[3] A. Lunardi: *Schauder theorems for linear elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^N$*, Studia Math. **128** (1998) 171–198.

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