Solutions to the Exercises from Lecture 4

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Exercise 4.1

Prove that $\mathrm{BUC}(\mathbb{R}^d)$ is the closure of $C^1_{\mathrm{b}}(\mathbb{R}^d)$ in $C_{\mathrm{b}}(\mathbb{R}^d)$.

Proof. Firstly, we will show that the $\mathrm{BUC}(\mathbb{R}^d)$ space is closed with respect to the sup-norm. Let f be a limit point of $\mathrm{BUC}(\mathbb{R}^d)$. Then, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{BUC}(\mathbb{R}^d)$ such that $f_n \to f$ uniformly. As $\mathrm{BUC}(\mathbb{R}^d) \subset C_{\mathrm{b}}(\mathbb{R}^d)$, and the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly, we know that f is bounded and continuous, so all that needs to be done to show that $\mathrm{BUC}(\mathbb{R}^d)$ is closed is to prove that f is uniformly continuous.

Let $\epsilon > 0$ be given. By the uniform convergence, there exists an $N \in \mathbb{N}$ such that

$$\|f_N - f\|_\infty = \sup\left\{|f_N(x) - f(x)| : x \in \mathbb{R}^d\right\} < \epsilon/3.$$

As the function f_N is (bounded and) uniformly continuous, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}^d$ with $|x - y| < \delta$ we have that $|f_N(x) - f_N(y)| < \epsilon/3$. For this δ, we find by the triangle inequality, for all $x, y \in \mathbb{R}^d$ with $|x - y| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \\ &\le |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows that f is uniformly continuous, so $\mathrm{BUC}(\mathbb{R}^d)$ is closed.

Now, we will show that for every element $f \in \mathrm{BUC}(\mathbb{R}^d)$ there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset C^1_{\mathrm{b}}(\mathbb{R}^d)$ such that $g_k \to f$ in sup-norm. For this, look at the "standard mollifier" on page 2 below, which is taken from Evans' *Partial Differential Equations*, [1] Appendix C.4 (page 629).

For every $k \in \mathbb{N}$, define the function g_k by

$$g_k(x) := \left(\eta_{(1/k)} * f\right)(x) = \int_{\mathbb{R}^d} \eta_{(1/k)}(y) f(x - y)\ \mathrm{d}y.$$

Let $k \in \mathbb{N}$ be arbitrary. Now, note that, for every $x \in \mathbb{R}^d$,

$$|g_k(x)| = \left|\int_{\mathbb{R}^d} \eta_{(1/k)}(y) f(x-y)\ \mathrm{d}y\right| \le \int_{\mathbb{R}^d} \left|\eta_{(1/k)}(y) f(x-y)\right|\ \mathrm{d}y \le \|f\|_\infty \int_{\mathbb{R}^d} \eta_{(1/k)}(y)\ \mathrm{d}y = \|f\|_\infty,$$

as f is uniformly bounded, $\eta_{(1/k)}$ is non-negative and $\int_{\mathbb{R}^d} \eta_{(1/k)}(y)\ \mathrm{d}y = 1$. This shows that $\|g_k\|_\infty \le \|f\|_\infty < \infty$ for every $k \in \mathbb{N}$.

We will proceed by showing that $\|g_k - f\|_\infty \to 0$ as $k \to \infty$, followed by the proof that, indeed, $g_k \in C^1_{\mathrm{b}}(\mathbb{R}^d)$ for all $k \ge 1$.

Let $\epsilon > 0$. As f is a (bounded and) uniformly continuous function, there exists a $\delta > 0$ such that for x, y with $|x - y| < \delta$ we have that $|f(x) - f(y)| < \epsilon/2$. Let $K \in \mathbb{N}$ be large enough to ensure that $1/K < \delta$. Then, for every $k \ge K$, we have that

$$\begin{aligned} |g_k(x) - f(x)| &= \left|\int_{\mathbb{R}^d} \eta_{(1/k)}(y) f(x-y)\ \mathrm{d}y - f(x)\right| = \left|\int_{\mathbb{R}^d} \eta_{(1/k)}(y) f(x-y)\ \mathrm{d}y - f(x)\int_{\mathbb{R}^d} \eta_{(1/k)}(y)\ \mathrm{d}y\right| \\ &= \left|\int_{\mathbb{R}^d} f(x-y)\eta_{(1/k)}(y)\ \mathrm{d}y - \int_{\mathbb{R}^d} f(x)\eta_{(1/k)}(y)\ \mathrm{d}y\right| \\ &\le \int_{\mathbb{R}^d} |f(x-y) - f(x)|\, \eta_{(1/k)}(y)\ \mathrm{d}y. \end{aligned}$$

DEFINITIONS. (i) *Define* $\eta \in C^\infty(\mathbb{R}^n)$ *by*

$$\eta(x) := \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

the constant $C > 0$ *selected so that* $\int_{\mathbb{R}^n} \eta \, dx = 1$.

(ii) *For each* $\epsilon > 0$, *set*

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right).$$

We call η the *standard mollifier*. The functions η_ϵ are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\epsilon \, dx = 1, \ \operatorname{spt}(\eta_\epsilon) \subset B(0, \epsilon).$$

DEFINITION. *If* $f : U \to \mathbb{R}$ *is locally integrable, define its* mollification

$$f^\epsilon := \eta_\epsilon * f \quad \text{in } U_\epsilon.$$

That is,

$$f^\epsilon(x) = \int_U \eta_\epsilon(x-y) f(y)\, dy = \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x-y)\, dy$$

for $x \in U_\epsilon$.

Figure 1: "Standard mollifier" from Evans' *Partial Differential Equations*, [1].

Now note that the support of $\eta_{(1/k)}$ is contained in $B(0,1/k)$, hence, the function $\eta_{(1/k)}$ is zero on $B(0,1/k)^C$, the complement of $B(0,1/k)$. This means that

$$\int_{\mathbb{R}^d} |f(x-y)-f(x)|\,\eta_{(1/k)}(y)\ \mathrm{d}y = \int_{B(0,1/k)} |f(x-y)-f(x)|\,\eta_{(1/k)}(y)\ \mathrm{d}y + \int_{B(0,1/k)^C} |f(x-y)-f(x)|\,\eta_{(1/k)}(y)\ \mathrm{d}y$$

and the second integral is zero by the remark we just made. When we look at the first integral, we have that $y \in B(0,1/k)$, i.e., $|y| < \frac{1}{k} \leq \frac{1}{K} < \delta$, which means that we also have that $|x-(x-y)| = |y| < \delta$ for every $x \in \mathbb{R}^d$ and therefore,

$$|f(x-y)-f(x)| < \frac{\epsilon}{2},$$

which means that

$$\int_{B(0,1/k)} |f(x-y)-f(x)|\,\eta_{(1/k)}(y)\ \mathrm{d}y \leq \frac{\epsilon}{2}\cdot\int_{B(0,1/k)} \eta_{(1/k)}(y)\ \mathrm{d}y \leq \frac{\epsilon}{2}\cdot\int_{\mathbb{R}^d} \eta_{(1/k)}(y)\ \mathrm{d}y = \frac{\epsilon}{2}.$$

Hence, for any $k \geq K$, we have

$$|g_k(x)-f(x)| \leq \frac{\epsilon}{2}$$

for every $x \in \mathbb{R}^d$. From this, it follows that, for every $k \geq K$,

$$\|g_k - f\|_\infty = \sup\left\{|g_k(x)-f(x)| : x \in \mathbb{R}^d\right\} \leq \frac{\epsilon}{2} < \epsilon,$$

proving that $\|g_k - f\|_\infty \to 0$, as $\epsilon > 0$ was arbitrary.

To see that g_k belongs to $C^1_\mathrm{b}(\mathbb{R}^d)$ for every $k \in \mathbb{N}$, we also need to establish that the function is differentiable and that its derivative is bounded. Let k be arbitrary. In this case, we have for every $x \in \mathbb{R}^d$,

$$g_k(x) := \left(\eta_{(1/k)} * f\right)(x) = \int_{\mathbb{R}^d} \eta_{(1/k)}(y) f(x-y)\ \mathrm{d}y.$$

Applying the coordinate transformation $z = x-y$, we still integrate over $\mathbb{R}^d$ and the absolute value of the determinant of the Jacobian matrix is one. Hence,

$$\int_{\mathbb{R}^d} \eta_{(1/k)}(y) f(x-y)\ \mathrm{d}y = \int_{\mathbb{R}^d} \eta_{(1/k)}(x-z) f(z)\ \mathrm{d}z =: \left(f * \eta_{(1/k)}\right)(x),$$

for every x, so $g_k = f * \eta_{(1/k)}$. Now, from the properties of mollifiers, as found in [2], we know from Theorem 2 therein, as f is indeed locally L^1 (it is uniformly bounded) and $\eta_{(1/k)}$ is C^∞ with compact support (see Figure 1), that g_k is $C^\infty(\mathbb{R}^d)$, so it is in particular $C^1(\mathbb{R}^d)$ (this can also be verified by hand using the Dominated Convergence Theorem). To show that it indeed belongs to $C^1_\mathrm{b}(\mathbb{R}^d)$ (the definition is not given in the lecture notes, but can be found in subsection E.1.1 in [3]), it suffices to show that all the partial derivatives of g_k are bounded. Let $i \in \{1,\ldots,d\}$ be arbitrary. Applying Lemma 7 in [2] to $g_k = f * \eta_{(1/k)}$, we find that

$$\frac{\partial}{\partial x_i} g_k(x) = \left(f * \frac{\partial}{\partial x_i}\eta_{(1/k)}\right)(x) = \int_{\mathbb{R}^d} f(z)\left(\frac{\partial}{\partial x_i}\eta_{(1/k)}(x-z)\right)\ \mathrm{d}z.$$

By using the transformation $y \mapsto x-z$, again the absolute value of the determinant of the Jacobian matrix is one, and we have

$$\int_{\mathbb{R}^d} f(z)\left(\frac{\partial}{\partial x_i}\eta_{(1/k)}(x-z)\right)\ \mathrm{d}z = \int_{\mathbb{R}^d} f(x-y)\left(\frac{\partial}{\partial x_i}\eta_{(1/k)}(y)\right)\ \mathrm{d}y.$$

For v with $|v| > 1/k$, we have that $\frac{\partial}{\partial x_i}\eta_{(1/k)}(v) = 0$. As $v \in \left\{y \in \mathbb{R}^d : |y| > 1/k\right\}$ and this set is open, we know that there exists an open ball $B_\mathrm{open}(v,\epsilon)$ with radius $\epsilon > 0$ around v such that $B_\mathrm{open}(v,\epsilon) \subset \left\{y \in \mathbb{R}^d : |y| > 1/k\right\}$. If we denote the i^th unit vector with $\vec{e}_i$, for all h small enough, we have that $v + h\vec{e}_i \in B_\mathrm{open}(v,\epsilon)$ and so

$$\eta_{(1/k)}(v + h\vec{e}_i) = 0$$

as supp $\left(\eta_{(1/k)}\right) \subset B(0,1/k)$. Since $\eta_{(1/k)}(v) = 0$ by the same argument, we find that

$$\frac{\eta_{(1/k)}(v + h\vec{e}_i) - \eta_{(1/k)}(v)}{h} = \frac{0}{h} = 0$$

for all small enough h, hence $\frac{\partial}{\partial x_i}\eta_{(1/k)}(v) = 0$. This means that, if $\frac{\partial}{\partial x_i}\eta_{(1/k)}(v) \neq 0$, that $|v| \leq 1/k$. Hence,

$$\left\{ v \in \mathbb{R}^d : \frac{\partial}{\partial x_i}\eta_{(1/k)}(v) \neq 0 \right\} \subset \left\{ y \in \mathbb{R}^d : |y| \leq 1/k \right\}$$

and so, we also have that

$$\operatorname{supp}\left(\frac{\partial}{\partial x_i}\eta_{(1/k)}\right) = \overline{\left\{ v \in \mathbb{R}^d : \frac{\partial}{\partial x_i}\eta_{(1/k)}(v) \neq 0 \right\}}^{\|\cdot\|_\infty} \subset \overline{\left\{ y \in \mathbb{R}^d : |y| \leq 1/k \right\}}^{\|\cdot\|_\infty} = \left\{ y \in \mathbb{R}^d : |y| \leq 1/k \right\} =: A.$$

Now we observe that $\frac{\partial \eta_{(1/k)}}{\partial x_i}(x) = k^{d-1}\frac{\partial \eta}{\partial x_i}(kx)$. We know that $\frac{\partial \eta}{\partial x_i}$ is bounded, by say M. This means that

$$\begin{aligned}
\left|\frac{\partial}{\partial x_i} g_k(x)\right| &\leq \int_{\mathbb{R}^d} \left| f(x-y)\left(\frac{\partial}{\partial x_i}\eta_{(1/k)}(y)\right)\right| \,\mathrm{d}y \leq \|f\|_\infty \int_{\mathbb{R}^d} \left|\left(\frac{\partial}{\partial x_i}\eta_{(1/k)}(y)\right)\right| \,\mathrm{d}y \\
&= \|f\|_\infty \left(\int_A \left|\left(\frac{\partial}{\partial x_i}\eta_{(1/k)}(y)\right)\right| \,\mathrm{d}y + \underbrace{\int_{A^c} \left|\left(\frac{\partial}{\partial x_i}\eta_{(1/k)}(y)\right)\right| \,\mathrm{d}y}_{=0} \right) \\
&\leq \|f\|_\infty \cdot M \cdot k^{d+1} \cdot \int_A 1 \,\mathrm{d}y = \|f\|_\infty \cdot M \cdot k \cdot \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}.
\end{aligned}$$

This shows that

$$\left|\frac{\partial}{\partial x_i} g_k(x)\right| \leq \|f\|_\infty \cdot M \cdot k \cdot \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$$

for every $x \in \mathbb{R}^d$. As i was arbitrary, all partial derivative of g_k are bounded. Hence, $\{g_k : k \geq 1\} \subset C_b^1(\mathbb{R}^d)$.

Now, we have the following inclusion and equality of sets,

$$\mathrm{BUC}(\mathbb{R}^d) \subset \overline{C_b^1(\mathbb{R}^d)}^{\|\cdot\|_\infty}, \quad \overline{\mathrm{BUC}(\mathbb{R}^d)}^{\|\cdot\|_\infty} = \mathrm{BUC}(\mathbb{R}^d),$$

and note that, by the Mean Value Theorem, every (bounded) function with a bounded derivative is uniformly continuous (and bounded), hence

$$C_b^1(\mathbb{R}^d) \subset \mathrm{BUC}(\mathbb{R}^d), \quad \text{which implies that} \quad \overline{C_b^1(\mathbb{R}^d)}^{\|\cdot\|_\infty} \subset \overline{\mathrm{BUC}(\mathbb{R}^d)}^{\|\cdot\|_\infty}.$$

Combining these inclusions and equalities, we obtain

$$\overline{\mathrm{BUC}(\mathbb{R}^d)}^{\|\cdot\|_\infty} = \mathrm{BUC}(\mathbb{R}^d) \subset \overline{C_b^1(\mathbb{R}^d)}^{\|\cdot\|_\infty} \subset \overline{\mathrm{BUC}(\mathbb{R}^d)}^{\|\cdot\|_\infty}, \text{ hence, } \mathrm{BUC}(\mathbb{R}^d) = \overline{C_b^1(\mathbb{R}^d)}^{\|\cdot\|_\infty}$$

and this shows that $\mathrm{BUC}(\mathbb{R}^d)$ is the closure of $C_\mathrm{b}^1(\mathbb{R}^d)$ in $C_\mathrm{b}(\mathbb{R}^d)$ (with respect to the sup-norm). □

References

[1] Lawrence C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 1998

[2] *Mollifiers and Smooth Functions* `https://andromeda.rutgers.edu/~loftin/ra1fal10/mollifier.pdf`, [Online; accessed 20-Dec-2016]

[3] Luca Lorenzi and Marcello Bertoldi, *Analytical Methods for Markov Semigroups*, Chapman & Hall/CRC, Boca Raton, Florida, 2007

Exercise 4.2

Prove that the function $z \mapsto T(z)f$, defined in (4.10) is holomorphic in the sector Σ_θ for every $\theta \in (0, \pi/2)$.

Clearly $K(z,x) = (4\pi z)^{-d/2} e^{-|x|^2/(4z)}$ is analytic in $\mathbb{C} \setminus \{0\}$. And since for all $k \in \mathbb{N}$, $x \in \mathbb{R}$ and $z \in \Sigma_\theta$ the function $y \mapsto \frac{\partial^k}{\partial z^k} K(z, x-y)$ can be written as the product of a polynomial in y and the mapping $y \mapsto e^{-|x-y|^2/(4z)}$, we may conclude that $y \mapsto \frac{\partial^k}{\partial z^k} K(z, x-y)$ is bounded and integrable with respect to y, for all $k \in \mathbb{N}$, $x \in \mathbb{R}$, $z \in \Sigma_{\pi/2}$. So by the dominated convergence theorem

$$Af(x) := \frac{\partial}{\partial z} T(z)f(x) = \frac{\partial}{\partial z} \int_{\mathbb{R}^d} K(z, x-y) f(y) \, \mathrm{d}y = \int_{\mathbb{R}^d} \frac{\partial}{\partial z} K(z, x-y) f(y) \, \mathrm{d}y.$$

So for all $x \in \mathbb{R}^d$

$$\begin{aligned}
\left| \frac{T(z+h)f(x) - T(z)f(x)}{h} - \int_{\mathbb{R}^d} \frac{\partial K(z,y)}{\partial z} f(x-y) \, \mathrm{d}y \right| &= \left| \int_{\mathbb{R}^d} \frac{K(z+h,y) - K(z,y)}{h} f(x-y) \, \mathrm{d}y - \int_{\mathbb{R}^d} \frac{\partial K(z,y)}{\partial z} f(x-y) \, \mathrm{d}y \right| \\
&= \left| \int_{\mathbb{R}^d} \frac{K(z,y) + h\frac{\partial K(z,y)}{\partial z} + \frac{1}{2}\frac{\partial^2 K(\sigma,y)}{\partial z^2} h^2 - K(z,y) - h\frac{\partial K(z,y)}{\partial z}}{h} \, \mathrm{d}y \right| \\
&\leq \int_{\mathbb{R}^d} \left| \frac{\frac{\partial^2 K(\sigma,y)}{\partial z^2} h^2}{2h} \right| |f(y-x)| \, \mathrm{d}y \leq M_1 M_2 |h| \to 0.
\end{aligned}$$

Where M_1 the bound of $f \in C_b(\mathbb{R}^d)$ and $M_2 = \int_{\mathbb{R}^d} \left| \frac{\partial^2 K(\sigma,y)}{\partial z^2} \right| \mathrm{d}y < \infty$. So we have shown that for all $z \in \Sigma_\theta$

$$\lim_{h \to 0} \left\| \frac{T(z+h)f - T(z)f}{h} - Af \right\|_\infty = 0.$$

This shows that $T(z)$ is holomorphic in the sector Σ_θ.

Exercise 4.3

Prove that the function

$$u(t,x) = \int_{\mathbb{R}^d} K(t, x-y) f(y) \, \mathrm{d}y, \quad t > 0, x \in \mathbb{R},$$

belongs to $C^\infty((0,+\infty) \times \mathbb{R}^d)$ for each $f \in C_b(\mathbb{R}^d)$.

We can prove by induction that for all $k \in \mathbb{N}$ and $t \in (0, \infty)$ there exists a polynomial $p_{t,k}$ of degree $2k$ such that

$$\frac{\partial^k}{\partial t^k} K(t,x) = p_{t,k}(x) e^{-|x|^2/(4t)}.$$

It follows that for all $t \in (0,\infty)$ and all $k \in \mathbb{N}$ the mapping $x \mapsto \frac{\partial^k}{\partial t^k} K(t,x)$ is bounded and integrable, and by the dominated convergence theorem it follows that

$$\frac{\partial^k}{\partial t^k} u(t,x) = \frac{\partial^k}{\partial t^k} \int_{\mathbb{R}^d} K(t, x-y) f(y) \, \mathrm{d}y = \int_{\mathbb{R}^d} \frac{\partial^k K(t, x-y)}{\partial t^k} f(y) \, \mathrm{d}y$$

A similar reasoning may be applied to the derivatives with respect to x_i, $i \in \{1, \ldots, d\}$.

Exercise 4.4

Note that it was demonstrated in Remark 4.1.7 that the sectorial operator associated with the Gauss-Weierstrass semigroup is an extension of the operator $(\Delta, C_b^2(\mathbb{R}^d))$. Thus in order to prove that for $d = 1$ it is in fact *equal* to $(\Delta, C_b^2(\mathbb{R}^d))$ it suffices to prove that $D(A) = C_b^2(\mathbb{R})$. To this end consider the following PDE:

$$u'' - u = g, \tag{1}$$

with $g \in C_b(\mathbb{R})$. One may check that if we define

$$u(x) = -\frac{1}{2} \int_{-\infty}^{x} e^{-x+s} g(s) \, ds + \frac{1}{2} \int_{x}^{\infty} e^{x-s} g(s) \, ds, \quad x \in \mathbb{R},$$

then $u \in C^2(\mathbb{R})$ and u satisfies (1). By a change of variables we see that

$$u(x) = \frac{1}{2}\int_0^\infty e^{-u}\left(g(u+x)+g(u-x)\right)du,$$

whence it immediately follows that $\|u\|_{C_b(\mathbb{R})} \leq \|g\|_{C_b(\mathbb{R})}$. In other words, $1 \in \rho(\Delta)$ (the resolvent set of Δ). Hence (recalling that A and Δ coincide on $C_b^2(\mathbb{R})$) we have that

$$(I-A)D(A) = C_b(\mathbb{R}) = (I-\Delta)C_b^2(\mathbb{R}) = (I-A)C_b^2(\mathbb{R}).$$

Applying $R(1,A)$ on both sides of the equation above we obtain $D(A) = C_b^2(\mathbb{R})$.

Exercise 4.5

Prove properties (iv), (v) and (vi) in Proposition 4.1.9.

Proof. (*iv*) Continuing from Lemma 4.1.2. (iv), we compute

$$\begin{aligned} D_{ijk}K(t,x) &= \left(\frac{-x_ix_jx_k}{8t^3} + \frac{x_i\delta_{jk}}{4t^2} + \frac{x_j\delta_{ik}}{4t^2} + \frac{x_k\delta_{ij}}{4t^2}\right)K(t,x) \\ &= \left(\frac{-x_ix_jx_k}{8t^3} + \frac{\langle x,\delta\rangle}{4t^2}\right)K(t,x), \end{aligned}$$

where $\langle x,\delta\rangle := x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij}$ to simplify notation. Fix $f \in C_b(\mathbb{R}^d)$. Applying the above result and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |(D_{ijk}T(t)f)(x)|^2 &= \left|\int_{\mathbb{R}^d}\left(\frac{-y_iy_jy_k}{8t^3} + \frac{\langle y,\delta\rangle}{4t^2}\right)K(t,y)f(x-y)\,\mathrm{d}y\right|^2 \\ &\leq \int_{\mathbb{R}^d} K(t,y)(f(t,x-y))^2\,\mathrm{d}y \int_{\mathbb{R}^d}\left(\frac{-y_iy_jy_k}{8t^3} + \frac{\langle y,\delta\rangle}{4t^2}\right)^2 K(t,y)\,\mathrm{d}y \\ &\leq ||f||_\infty^2 \int_{\mathbb{R}^d}\left(\frac{-y_iy_jy_k}{8t^3} + \frac{\langle y,\delta\rangle}{4t^2}\right)^2 K(t,y)\,\mathrm{d}y \\ &= ||f||_\infty^2 \left(\frac{1}{64t^6}\int_{\mathbb{R}^d} y_i^2y_j^2y_k^2K(t,y)\,\mathrm{d}y \right. \\ &\quad - \frac{1}{16t^5}\int_{\mathbb{R}^d}\left(y_i^2y_jy_k\delta_{jk} + y_iy_j^2y_k\delta_{ik} + y_iy_jy_k^2\delta_{ij}\right)K(t,y)\,\mathrm{d}y \\ &\quad + \frac{1}{16t^4}\int_{\mathbb{R}^d}\left(y_i^2\delta_{jk} + y_j^2\delta_{ik} + y_k^2\delta_{ij}\right)K(t,y)\,\mathrm{d}y \\ &\quad \left. + \frac{1}{8t^4}\int_{\mathbb{R}^d}\left(y_iy_j\delta_{jk}\delta_{ik} + y_iy_k\delta_{jk}\delta_{ij} + y_jy_k\delta_{ik}\delta_{ij}\right)K(t,y)\,\mathrm{d}y\right). \end{aligned}$$

If $i=j=k$, then by (4.11) with $k=2,4,6$ and a being the i-th element of the Euclidean basis of $\mathbb{R}^d$, we get

$$\int_{\mathbb{R}^d} y_i^2K(t,y)\,\mathrm{d}y = 2t, \qquad \int_{\mathbb{R}^d} y_i^4K(t,y)\,\mathrm{d}y = 12t^2, \qquad \int_{\mathbb{R}^d} y_i^6K(t,y)\,\mathrm{d}y = 120t^3.$$

If $i \neq j$ and $i=k$, using the one-dimensional version of formula (4.11), we get

$$\begin{aligned} \int_{\mathbb{R}^d} y_i^4y_j^2K(t,y)\,\mathrm{d}y &= \left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} y_i^4e^{-\frac{y_i^2}{4t}}\,\mathrm{d}y_i\right)\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} y_j^2e^{-\frac{y_j^2}{4t}}\,\mathrm{d}y_j\right) \\ &\qquad \times \prod_{h\neq i,j}\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{y_h^2}{4t}}\,\mathrm{d}y_h\right) \\ &= 24t^3, \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^d} y_i^2y_j^2K(t,y)\,\mathrm{d}y &= \left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} y_i^2e^{-\frac{y_i^2}{4t}}\,\mathrm{d}y_i\right)\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} y_j^2e^{-\frac{y_j^2}{4t}}\,\mathrm{d}y_j\right) \\ &\qquad \times \prod_{h\neq i,j}\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{y_h^2}{4t}}\,\mathrm{d}y_h\right) \\ &= 4t^2. \end{aligned}$$

If $i \neq j$ and $j \neq k$ and $i \neq k$, then similarly we find

$$\int_{\mathbb{R}^d} y_i^2 y_j^2 y_k^2 K(t,y)\,\mathrm{d}y = \left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} y_i^2 e^{-\frac{y_i^2}{4t}}\,\mathrm{d}y_i\right)\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} y_j^2 e^{-\frac{y_j^2}{4t}}\,\mathrm{d}y_j\right)$$
$$\times\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} y_k^2 e^{-\frac{y_k^2}{4t}}\,\mathrm{d}y_k\right)\prod_{h\neq i,j}\left(\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{y_h^2}{4t}}\,\mathrm{d}y_h\right)$$
$$= 8t^3.$$

Hence,

$$|(D_{ijk}T(t)f)(x)|^2 \leq ||f||_\infty^2 \times \begin{cases} \frac{6}{8t^3} & \text{if } i=j=k \\ \frac{2}{8t^3} & \text{if } i\neq j \text{ and } i=k \\ \frac{1}{8t^3} & \text{if } i\neq j \text{ and } i\neq k \text{ and } j\neq k \end{cases}.$$

Property (iv) follows.

(v) As it is immediately seen, if $f \in C_b^1(\mathbb{R}^d)$, then $(D_iT(t)f)(x) = (T(t)D_if)(x)$ for $(t,x) \in [0,\infty)\times\mathbb{R}^d$, $i = 1,\dots,d$ and, consequently, thanks to property (ii), we can estimate

$$||D_{ijk}T(t)f||_\infty = ||D_{ij}T(t)D_kf||_\infty \leq \frac{\sqrt{1+\delta_{ij}}}{2t}||D_kf||_\infty \leq \frac{\sqrt{1+\delta_{ij}}}{2t}||\nabla_x f||_\infty.$$

Similarly

$$||D_{ijk}T(t)f||_\infty \leq \frac{\sqrt{1+\delta_{jk}}}{2t}||\nabla_x f||_\infty.$$

and

$$||D_{ijk}T(t)f||_\infty \leq \frac{\sqrt{1+\delta_{ik}}}{2t}||\nabla_x f||_\infty.$$

Property (v) follows.

(vi) As it is immediately seen, if $f \in C_b^2(\mathbb{R}^d)$, then $(D_{ij}T(t)f)(x) = (T(t)D_{ij}f)(x)$ for $(t,x) \in [0,\infty)\times\mathbb{R}^d$, $i,j = 1,\dots,d$ and, consequently, thanks to property (i), we can estimate

$$||D_{ijk}T(t)f||_\infty = ||D_iT(t)D_{jk}f||_\infty \leq ||\nabla_x T(t)D_{jk}f||_\infty \leq \frac{1}{\sqrt{2t}}||D_{jk}f||_\infty \leq \frac{1}{\sqrt{2t}}||D^2f||_\infty.$$

□

Exercise 4.6

Use (4.9) and Proposition 3.2.8 *to prove that the operator*

$$A + b(\cdot)\nabla \quad \text{with domain} \quad D(A)$$

generates an analytic semigroup in $C_b(\mathbb{R}^d)$, where $b \in C_b(\mathbb{R}^d,\mathbb{R}^d)$ and A is the generator of the Gauss-Weierstrass semigroup $\{T(t)\}$ in $C_b(\mathbb{R}^d)$.

Proof. For ease of notation write $B = b(\cdot)\nabla$. If we assume $||BR(\lambda,A)|| < 1$ for $Re\lambda > \lambda_0$ for some $\lambda_0 \in \mathbb{R}$, then the operator $(I - BR(\lambda,A))$ is invertible, as well as the operator

$$(\lambda I - A) - B = (I - BR(\lambda,A))(\lambda I - A),$$

and

$$R(\lambda, A+B) = R(\lambda,A)(I - BR(\lambda,A))^{-1} = R(\lambda,A)\sum_{k=0}^{\infty}[BR(\lambda,A)]^k$$

(see Lemma 2.1.11). Then the resolvent set of $A+B$ contains an half-plane. Moreover $\|\lambda R(\lambda,A+B)\| \leq \|\lambda R(\lambda,A)\|C \leq MC$. So we want to show that $||BR(\lambda,A)|| < 1$ for $Re\lambda > \lambda_0$. Note that $||BR(\lambda,A)|| \leq \|b\|||\nabla R(\lambda,A)||$. By the gradient estimates of the Gauss-Weierstrass semigroup we have

$$\|\nabla R(\lambda,A)f\| = \left\|\nabla\int_0^\infty e^{-\lambda t}T(t)f dt\right\| \leq \int_0^\infty e^{-Re\lambda t}\|\nabla T(t)f\|dt$$
$$\leq \|f\|\int_0^\infty \frac{e^{-Re\lambda t}}{\sqrt{2t}}dt = \frac{\|f\|}{\sqrt{2Re\lambda}}\int_0^\infty \frac{e^{-s}}{\sqrt{s}}ds.$$

For $Re\lambda > \frac{1}{2}\left(\int_0^\infty \frac{e^{-s}}{\sqrt{s}}ds\right)^2$ we have the desired result.

□

Exercise 4.7

Prove that for each $\theta \in (2,3)$ there exists a positive C_θ such that

$$\|f\|_{C_\mathrm{b}^\theta(\mathbb{R}^d)} \leq C_\theta \|f\|_{C_\mathrm{b}^3(\mathbb{R}^d)}^{\theta-2} \|f\|_{C_\mathrm{b}^2(\mathbb{R}^d)}^{3-\theta}$$

for every $f \in C_\mathrm{b}^3(\mathbb{R}^d)$. [This assumption is misprinted in the Lecture Notes and is necessary for the $\|\cdot\|_{C_\mathrm{b}^3(\mathbb{R}^d)}$-norm to exist; compare this with Lemma 4.1.11, where we take $f \in C_\mathrm{b}^1(\mathbb{R}^d)$.]

Proof. First, note that $\lfloor\theta\rfloor = 2$, so

$$\|f\|_{C_\mathrm{b}^\theta(\mathbb{R}^d)} = \sum_{|\beta|\leq 2} \left\| \frac{\partial^{|\beta|} f}{\partial x^\beta} \right\|_\infty + \sum_{|\beta|=2} \left[\frac{\partial^2 f}{\partial x^\beta} \right]_{C^{\theta-\lfloor\theta\rfloor}(\mathbb{R}^d)}.$$

Note that, by the forum post "Theorem 4.1.12", we find that

$$\sum_{|\beta|\leq 2} \left\| \frac{\partial^{|\beta|} f}{\partial x^\beta} \right\|_\infty = \|f\|_{C_\mathrm{b}^2(\mathbb{R}^d)} \quad \text{and} \quad \sum_{|\beta|\leq 2} \left\| \frac{\partial^{|\beta|} f}{\partial x^\beta} \right\|_\infty \leq \|f\|_{C_\mathrm{b}^3(\mathbb{R}^d)}.$$

This means that, as $1 = (\theta-2)+(3-\theta)$

$$\sum_{|\beta|\leq 2} \left\| \frac{\partial^{|\beta|} f}{\partial x^\beta} \right\|_\infty = \left(\sum_{|\beta|\leq 2} \left\| \frac{\partial^{|\beta|} f}{\partial x^\beta} \right\|_\infty \right)^{\theta-2} \left(\sum_{|\beta|\leq 2} \left\| \frac{\partial^{|\beta|} f}{\partial x^\beta} \right\|_\infty \right)^{3-\theta} \leq \|f\|_{C_\mathrm{b}^3(\mathbb{R}^d)}^{\theta-2} \|f\|_{C_\mathrm{b}^2(\mathbb{R}^d)}^{3-\theta}.$$

Moreover, note that, for all $x, y \in \mathbb{R}^d$,

$$\left| \frac{\partial^2 f}{\partial x^\beta}(x) - \frac{\partial^2 f}{\partial x^\beta}(y) \right| \leq \left| \frac{\partial^2 f}{\partial x^\beta}(x) \right| + \left| \frac{\partial^2 f}{\partial x^\beta}(y) \right| \leq 2 \left\| \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty \leq 2\|f\|_{C_\mathrm{b}^2(\mathbb{R}^d)}$$

and that

$$\left| \frac{\partial^2 f}{\partial x^\beta}(x) - \frac{\partial^2 f}{\partial x^\beta}(y) \right| \leq \left\| \nabla \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty |x-y|,$$

by similar arguments as in Lemma 4.1.12. We have that

$$\left\| \nabla \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty = \sup\left\{ \left\| \nabla \frac{\partial^2 f}{\partial x^\beta}(a) \right\|_{\mathbb{R}^d} : a \in \mathbb{R}^d \right\} = \sup\left\{ \left(\sum_{i=1}^d \left| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta}(a) \right|^2 \right)^{\frac{1}{2}} : a \in \mathbb{R}^d \right\}.$$

and

$$\begin{aligned} \sup\left\{ \left(\sum_{i=1}^d \left| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta}(a) \right|^2 \right)^{\frac{1}{2}} : a \in \mathbb{R}^d \right\} &= \left(\sup\left\{ \sum_{i=1}^d \left| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta}(a) \right|^2 : a \in \mathbb{R}^d \right\} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^d \sup\left\{ \left| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta}(a) \right|^2 : a \in \mathbb{R}^d \right\} \right)^{\frac{1}{2}}. \end{aligned}$$

Furthermore, we have

$$\sup\left\{ \left| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta}(a) \right|^2 : a \in \mathbb{R}^d \right\} = \left(\sup\left\{ \left| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta}(a) \right| : a \in \mathbb{R}^d \right\} \right)^2 = \left\| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty^2$$

and by the subaddivity of the square root function, we obtain

$$\begin{aligned} \left(\sum_{i=1}^d \sup\left\{ \left| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta}(a) \right|^2 : a \in \mathbb{R}^d \right\} \right)^{\frac{1}{2}} &= \left(\sum_{i=1}^d \left\| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^d \left(\left\| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty^2 \right)^{\frac{1}{2}} \\ &= \sum_{i=1}^d \left\| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty. \end{aligned}$$

Observe that, for every $i \in \{1, \dots, d\}$,

$$\left\| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty = \left\| \frac{\partial^3 f}{\partial x^{\beta + \vec{e}_i}} \right\|_\infty$$

where $\vec{e}_i$ is i^{th} unit vector in $\mathbb{R}^d$. This shows that

$$\left\| \nabla \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty \leq \sum_{i=1}^d \left\| \frac{\partial}{\partial x_i} \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty \leq \|f\|_{C^3_{\mathrm{b}}(\mathbb{R}^d)} .$$

for every vector β with $|\beta| = 2$. Now, with the same arguments as in Lemma 4.1.11, we have

$$\begin{aligned} \left| \frac{\partial^2 f}{\partial x^\beta}(x) - \frac{\partial^2 f}{\partial x^\beta}(y) \right| &= \left| \frac{\partial^2 f}{\partial x^\beta}(x) - \frac{\partial^2 f}{\partial x^\beta}(y) \right|^{\theta - 2} \left| \frac{\partial^2 f}{\partial x^\beta}(x) - \frac{\partial^2 f}{\partial x^\beta}(y) \right|^{3 - \theta} \\ &\leq \left\| \nabla \frac{\partial^2 f}{\partial x^\beta} \right\|_\infty^{\theta - 2} |x - y|^{\theta - 2} \, 2^{3 - \theta} \, \|f\|_{C^2_{\mathrm{b}}(\mathbb{R}^d)}^{3 - \theta} \\ &\leq \|f\|_{C^3_{\mathrm{b}}(\mathbb{R}^d)}^{\theta - 2} |x - y|^{\theta - 2} \, 2^{3 - \theta} \, \|f\|_{C^2_{\mathrm{b}}(\mathbb{R}^d)}^{3 - \theta} \end{aligned}$$

so

$$\left[\frac{\partial^2 f}{\partial x^\beta} \right]_{C^{\theta - \lfloor \theta \rfloor}(\mathbb{R}^d)} \leq 2^{3 - \theta} \, \|f\|_{C^3_{\mathrm{b}}(\mathbb{R}^d)}^{\theta - 2} \, \|f\|_{C^2_{\mathrm{b}}(\mathbb{R}^d)}^{3 - \theta}$$

for every vector $\beta \in \mathbb{N}_0^d$ with $|\beta| = 2$.

Taking this all together, we obtain,

$$\begin{aligned} \|f\|_{C^\theta_{\mathrm{b}}(\mathbb{R}^d)} &= \sum_{|\beta| \leq 2} \left\| \frac{\partial^{|\beta|} f}{\partial x^\beta} \right\|_\infty + \sum_{|\beta| = 2} \left[\frac{\partial^2 f}{\partial x^\beta} \right]_{C^{\theta - \lfloor \theta \rfloor}(\mathbb{R}^d)} \\ &\leq \|f\|_{C^3_{\mathrm{b}}(\mathbb{R}^d)}^{\theta - 2} \, \|f\|_{C^2_{\mathrm{b}}(\mathbb{R}^d)}^{3 - \theta} + \sum_{|\beta| = 2} 2^{3 - \theta} \, \|f\|_{C^3_{\mathrm{b}}(\mathbb{R}^d)}^{\theta - 2} \, \|f\|_{C^2_{\mathrm{b}}(\mathbb{R}^d)}^{3 - \theta} \\ &= \left(1 + \sum_{|\beta| = 2} 2^{3 - \theta} \right) \|f\|_{C^3_{\mathrm{b}}(\mathbb{R}^d)}^{\theta - 2} \, \|f\|_{C^2_{\mathrm{b}}(\mathbb{R}^d)}^{3 - \theta} , \end{aligned}$$

so the statement holds with $C_\theta = \left(1 + \sum_{|\beta| = 2} 2^{3 - \theta} \right)$. □