Solutions to the exercises of Lecture 8 of the Internet Seminar 2016/17

Team Karlsruhe

Exercise 8.4.1

Assumption: Let $\alpha \in (0,1)$ and $\theta \in (0,1)$. We introduce the sets

$$X = \{u \in C^{1,2}((0,T] \times \mathbb{R}^d) \cap C_b^{0,2+\alpha-2\theta}([0,T] \times \mathbb{R}^d) : \\ \sup_{t \in (0,T]} t^\theta (\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}) < \infty\}$$

and

$$Y = C_b^{2+\alpha-2\theta}(\mathbb{R}^d) \times \{g \in C((0,T] \times \mathbb{R}^d) : \sup_{t \in (0,T]} t^\theta \|g(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} < \infty\},$$

endowed with the norms

$$\|u\|_X = \|u\|_{C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)} + \sup_{t \in (0,T]} t^\theta (\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)})$$

and

$$\|(f,g)\|_Y = \|f\|_{C_b^{2+\alpha-2\theta}(\mathbb{R}^d)} + \sup_{t \in (0,T]} t^\theta \|g(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)}.$$

Claim: The spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces.

Proof: It is clear that $(X, \|\cdot\|_X)$ and $(Y, \|, \cdot\|_Y)$ are normed vector spaces. We prove in the following that they are complete.

We first note that by setting

$$|||u|||_X := \sup_{t \in (0,T]} t^\theta \left(\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \right)$$

for $u \in X$ one defines a norm $|||\cdot|||_X$ on X.
Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, \|\cdot\|_X)$. Then the sequence $(u_n)_{n\in\mathbb{N}}$ is also Cauchy in $C_b^{0,2+\alpha-2\theta}([0,T]\times \mathbb{R}^d)$ and in $(X, |||\cdot|||_X)$. The completeness of $C_b^{0,2+\alpha-2\theta}([0,T] \times \mathbb{R}^d)$ gives a $u \in C_b^{0,2+\alpha-2\theta}([0,T] \times \mathbb{R}^d)$ such that $u_n \to u$ as $n \to \infty$ in $C_b^{0,2+\alpha-2\theta}([0,T] \times \mathbb{R}^d)$ and thus in particular with respect to the supremum norm on $[0,T] \times \mathbb{R}^d$. We estimate for each $\delta \in (0,T)$ that

$$\begin{aligned}
&\|u_n - u_m\|_{C_b^{1,2}([\delta,T]\times\mathbb{R}^d)} \\
&\quad \le \delta^{-\theta} \sup_{t \in [\delta,T]} t^\theta \left(\|u_n(t,\cdot) - u_m(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)} + \|D_t u_n(t,\cdot) - D_t u_m(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \right) \\
&\quad \le \delta^{-\theta} |||u_n - u_m|||_X,
\end{aligned}$$

so that $(u_n)_{n\in\mathbb{N}}$ is Cauchy in $C_b^{1,2}([\delta,T]\times\mathbb{R}^d)$. Since $C_b^{1,2}([\delta,T]\times\mathbb{R}^d)$ is complete, we hence get for each $\delta\in(0,T)$ a $u_\delta\in C_b^{1,2}([\delta,T]\times\mathbb{R}^d)$ with $u_n\to u_\delta$ as $n\to\infty$ in $C^{1,2+\alpha}([\delta,T]\times\mathbb{R}^d)$ and thus in particular with respect to the supremum norm on $[\delta,T]\times\mathbb{R}^d$. Since uniform limits are unique, the restriction of u to $[\delta,T]\times\mathbb{R}^d$ coincides with u_δ for all $\delta\in(0,T)$, so that $u\in C^{1,2}((0,T]\times\mathbb{R}^d)\cap C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)$ and $u_n\to u$ in $C^{1,2}([\delta,T]\times\mathbb{R}^d)\cap C_b^{0,2+\alpha-2\theta}([0,T]\times\mathbb{R}^d)$ for all $\delta\in(0,T)$ as $n\to\infty$.
The boundedness of $(u_n)_{n\in\mathbb{N}}$ with respect to $|||\cdot|||_X$ (Cauchy sequence!) yields

$$\|u_n(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}\le t^{-\theta}\sup_{n\in\mathbb{N}}|||u_n|||_X \qquad\text{and}\qquad \|D_tu_n(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\le t^{-\theta}\sup_{n\in\mathbb{N}}|||u_n|||_X$$

for all $t\in(0,T]$. These uniform bounds on the Hölder norms give us together with the pointwise convergence of $u_n(t,\cdot)$ to $u(t,\cdot)$ as $n\to\infty$ that $u_n(t,\cdot)\to u(t,\cdot)$ in $C_b^{2+\alpha}(\mathbb{R}^d)$ and $D_tu_n(t,\cdot)\to D_tu(t,\cdot)$ in $C_b^{\alpha}(\mathbb{R}^d)$ as $n\to\infty$ and the estimates

$$\|u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}\le t^{-\theta}\sup_{n\in\mathbb{N}}|||u_n|||_X \qquad\text{and}\qquad \|D_tu(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\le t^{-\theta}\sup_{n\in\mathbb{N}}|||u_n|||_X$$

for all $t\in(0,T]$. We thus conclude

$$|||u|||_X\le 2\sup_{n\in\mathbb{N}}|||u_n|||_X<\infty.$$

Let $\varepsilon>0$. Then there exists an $n_\varepsilon\in\mathbb{N}$ so large that $|||u_n-u_m|||_X\le\varepsilon$ for all $n,m\ge n_\varepsilon$, since $(u_n)_{n\in\mathbb{N}}$ is Cauchy with respect to $|||\cdot|||_X$. For all $n\ge n_\varepsilon$ this gives us

$$\begin{aligned}
&t^\theta\left(\|u_n(t,\cdot)-u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}+\|D_tu_n(t,\cdot)-D_tu(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\right)\\
&\quad=\lim_{m\to\infty}t^\theta\left(\|u_n(t,\cdot)-u_m(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}+\|D_tu_n(t,\cdot)-D_tu_m(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\right)\\
&\quad\le\limsup_{m\to\infty}\sup_{t\in(0,T]}t^\theta\left(\|u_n(t,\cdot)-u_m(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}+\|D_tu_n(t,\cdot)-D_tu_m(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\right)\\
&\quad=\limsup_{m\to\infty}|||u_n-u_m|||_X\le\varepsilon
\end{aligned}$$

for all $t\in(0,T]$, so that

$$|||u_n-u|||_X=\sup_{t\in(0,T]}t^\theta\left(\|u_n(t,\cdot)-u(t,\cdot)\|_{C_b^{2+\alpha}(\mathbb{R}^d)}+\|D_tu_n(t,\cdot)-D_tu(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\right)\le\varepsilon.$$

This shows that $u_n\to u$ with respect to $|||\cdot|||_X$ as $n\to\infty$. Summing up, we conclude $u_n\to u$ with respect to $\|\cdot\|_X$ as $n\to\infty$.

The argumentation for Y is similar than the one for X. We denote

$$|||h|||_{\widetilde{Y}}:=\sup_{t\in(0,T]}t^\theta\|h(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}$$

for $h\in\widetilde{Y}:=\{g\in C((0,T]\times\mathbb{R}^d):\sup_{t\in(0,T]}t^\theta\|g(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}<\infty\}$ and see that it is a norm on this vector space.
Let $(f_n,g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(Y,\|\cdot\|_Y)$. Then $(f_n)_{n\in\mathbb{N}}$ is Cauchy in $C_b^{2+\alpha-2\theta}(\mathbb{R}^d)$ and $(g_n)_{n\in\mathbb{N}}$ is Cauchy and thus bounded in $(\widetilde{Y},|||\cdot|||_{\widetilde{Y}})$. Since $C_b^{2+\alpha-2\theta}(\mathbb{R}^d)$ is a Banach space, there exists an $f\in C_b^{2+\alpha-2\theta}(\mathbb{R}^d)$ with $f_n\to f$ in $C_b^{2+\alpha-2\theta}(\mathbb{R}^d)$ as $n\to\infty$.
For every $\delta\in(0,T)$ we have, similar to above,

$$\sup_{t\in(\delta,T]}\|g_n(t,\cdot)-g_m(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\le\delta^{-\theta}\sup_{t\in(\delta,T]}t^\theta\|g_n(t,\cdot)-g_m(t,\cdot)\|_{C_b^{\alpha}(\mathbb{R}^d)}\le\delta^{-\theta}|||g_n-g_m|||_{\widetilde{Y}}$$

and hence $(g_n)_{n\in\mathbb{N}}$ is Cauchy in $C((\delta,T],C_b^{\alpha}(\mathbb{R}^d))$. Since $C((\delta,T],C_b^{\alpha}(\mathbb{R}^d))$ is complete, there exists a $g_\delta\in C((\delta,T],C_b^{\alpha}(\mathbb{R}^d))$ with $g_n\to g_\delta$ in $C((\delta,T],C_b^{\alpha}(\mathbb{R}^d))$ as $n\to\infty$. It is clear that for $0<\delta_1<\delta_2$ we have

$g_{\delta_2} = g_{\delta_1}|_{(\delta_2,T]}$ due to the uniqueness of uniform limits. This gives us that the setting $g(t,x) := g_\delta(t,x)$ for all $(t,x) \in (0,T] \times \mathbb{R}^d$ and $\delta \in (0,t)$ is independent of δ and thus well-defined. We infer $g \in C((0,T] \times \mathbb{R}^d)$ from the preservation of continuity by uniform limits.
Let $\varepsilon > 0$. Then there exists an $n_\varepsilon \in \mathbb{N}$ so large such that $|||g_n - g_m|||_{\widetilde{Y}} \leq \varepsilon$ for all $n, m \geq n_\varepsilon$ since $(g_n)_{n\in\mathbb{N}}$ is Cauchy with respect to $|||\cdot|||_{\widetilde{Y}}$. Thus, for all $n \geq n_\varepsilon$ there holds

$$\begin{aligned} t^\theta \|g(t,\cdot) - g_n(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} &= \lim_{m\to\infty} t^\theta \|g_m(t,\cdot) - g_n(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \leq \limsup_{m\to\infty} \sup_{t\in(0,T]} t^\theta \|g_m(t,\cdot) - g_n(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \\ &= \limsup_{m\to\infty} |||g_m - g_n|||_{\widetilde{Y}} \leq \varepsilon \end{aligned}$$

for all $t \in (0,T]$, so that

$$|||g - g_n|||_{\widetilde{Y}} = \sup_{t\in(0,T]} t^\theta \|g_t - g_n(t,\cdot)\|_{C_b^\alpha(\mathbb{R}^d)} \leq \varepsilon,$$

which shows that $g_n \to g$ as $n \to \infty$ with respect to $|||\cdot|||_{\widetilde{Y}}$ and hence that $|||g|||_{\widetilde{Y}} < \infty$.
Summing up, we have $(f,g) \in Y$ and $(f_n, g_n) \to (f,g)$ with respect to $\|\cdot\|_Y$ as $n \to \infty$.

Exercise 8.4.2

Assumption: Let $u \in C_b([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$ be such that $D_t u - \mathcal{A}u \leq 0$ on $(0,T] \times \mathbb{R}^d$ and $u(0,\cdot) \leq 0$ on $\mathbb{R}^d$, where $\mathcal{A}$ is the elliptic operator introduced at the beginning of Lecture 7.

(a) *Assumption:* We define the function $\varphi : \mathbb{R}^d \to \mathbb{R}$ by $\varphi(x) := 1 + |x|^2$.

Claim: There exists a $\lambda > 0$ such that $\mathcal{A}\varphi \leq \lambda\varphi$ on $\mathbb{R}^d$.

Proof: With

$$1 \leq \varphi(x) \qquad \text{and} \qquad x_j \leq \frac{1}{2} + \frac{1}{2}x_j^2 \leq \frac{1}{2}\varphi(x)$$

for all $x \in \mathbb{R}^d$ and $j = 1, \ldots, d$, we have

$$\begin{aligned} \mathcal{A}\varphi(x) &= \sum_{j=1}^d q_{jj}(x) \cdot 2 + \sum_{j=1}^d b_j(x) \cdot 2x_j + c(x)\varphi(x) \\ &\leq 2\sum_{j=1}^d \|q_{jj}\|_\infty \varphi(x) + \sum_{j=1}^d \|b_j\|_\infty \varphi(x) + \Big(\sup_{x\in\mathbb{R}^d} c(x)\Big)\varphi(x) \\ &= \Big(2\sum_{j=1}^d \|q_{jj}\|_\infty + \sum_{j=1}^d \|b_j\|_\infty + \sup_{x\in\mathbb{R}^d} c(x)\Big)\varphi(x) \end{aligned}$$

for all $x \in \mathbb{R}^d$, so that the statement holds true for all $\lambda \in \mathbb{R}$ with

$$\lambda \geq \lambda_0 := 2\sum_{j=1}^d \|q_{jj}\|_\infty + \sum_{j=1}^d \|b_j\|_\infty + \sup_{x\in\mathbb{R}^d} c(x).$$

(b) *Assumption:* We define for all $n \in \mathbb{N}$ the function $v_n : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ by $v_n(t,x) := u(t,x) - e^{(\lambda+1)t}\frac{1}{n}\varphi(x)$ with λ and φ from part (a).

Remark: The functions v_n we choose for the following proof differ from the ones suggested in the original problem.

Claim: For all $n \in \mathbb{N}$ we have $v_n \in C([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T] \times \mathbb{R}^d)$, v_n achieves its maximum on $[0,T] \times \mathbb{R}^d$ and v_n satisfies the differential inequality $D_t v_n - \mathcal{A}v_n < 0$ on $(0,T] \times \mathbb{R}^d$.

Remark: The claim in the original exercise that v_n is bounded on $[0,T]\times\mathbb{R}^d$ is not true, since u is bounded on $[0,T]\times\mathbb{R}^d$, but φ is unbounded on $\mathbb{R}^d$.

Proof: Let $n\in\mathbb{N}$. Due to the regularity of u it is clear that $v_n\in C([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T]\times\mathbb{R}^d)$. Choose $R:=\sqrt{n(2\|u\|_\infty+1)-1}$. We observe

$$v_n(0,0)=u(0,0)-\tfrac{1}{n}\geq-\|u\|_\infty-1$$

and

$$v_n(t,x)\leq\|u\|_\infty-\tfrac{1}{n}(1+R^2)\leq\|u\|_\infty-(2\|u\|_\infty+1)=-\|u\|_\infty-1$$

on $[0,T]\times(\mathbb{R}^d\setminus B(0,R))$, so that v_n has a global maximum, which is attained on the compact set $[0,T]\times\overline{B}(0,R)$. Additionally, we get on $(0,T]\times\mathbb{R}^d$ the estimate

$$\begin{aligned}
&D_tv_n(t,x)-\mathcal{A}v_n(t,x)\\
&\quad=D_tu(t,x)-\mathcal{A}u(t,x)-(\lambda+1)e^{(\lambda+1)t}\tfrac{1}{n}\varphi(x)+e^{(\lambda+1)t}\tfrac{1}{n}\mathcal{A}\varphi(x)\\
&\quad\leq(D_tu(t,x)-\mathcal{A}u(t,x))+(-(\lambda+1)+\lambda)e^{(\lambda+1)t}\tfrac{1}{n}\varphi(x)\\
&\quad=-e^{(\lambda+1)t}\tfrac{1}{n}\varphi(x)<0.
\end{aligned}$$

(c) *Claim:* For all $n\in\mathbb{N}$ we have $v_n\leq0$ and $u\leq0$ on $[0,T]\times\mathbb{R}^d$.

Proof: Let $c_0:=\|c\|_\infty$. Then the function $w:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ defined by $w(x,t):=e^{-c_0t}u(t,x)$ and the operator $\mathcal{A}-c_0I$ satisfy the assumptions and thus the assertions in (a) and (b), where we can take the same λ. In addition, $\mathcal{A}-c_0I$ has the potential $\widetilde{c}=c-c_0\leq0$. We show now claim (c) under the extra assumption $c\leq0$, so that $w\leq0$ and hence $v\leq0$.
Let $n\in\mathbb{N}$. If the maximum from part (b) is smaller or equal to 0, we have $v_n\leq0$. So, we assume that this maximum is strictly positive. Since $v_n(0,x)\leq u(x,0)\leq0$ on $\mathbb{R}^d$, there thus exists a $(t_0,x_0)\in(0,T]\times\mathbb{R}^d$ such that v_n has its maximum at (t_0,x_0). Then we have $D_tv_n(t_0,x_0)\geq0$ (since $t_0=T$ is possible), $D_jv_n(t_0,x_0)=0$ for all $j=1,\dots,d$ and the matrix $(D_{ij}v_n(t_0,x_0))_{i,j=1}^d$ is negative semi-definite. From Exercise 1.4.1 we thus conclude

$$D_tv_n(t_0,x_0)-\mathcal{A}v_n(t_0,x_0)=D_tv_n(t_0,x_0)-\sum_{i,j=1}^d q_{ij}(x_0)D_{ij}v_n(t_0,x_0)-c(x_0)v_n(t_0,x_0)\geq0,$$

which is a contradiction to the result $D_tv_n-\mathcal{A}v_n<0$ of part (b). Hence, we have $v_n\leq0$. Since $v_n(t,x)$ converges to $u(t,x)$ as $n\to\infty$ for all $(t,x)\in[0,T]\times\mathbb{R}^d$, it follows $u\leq0$ on $[0,T]\times\mathbb{R}^d$.

(d) *Assumption:* Let $f\in C_b(\mathbb{R}^d)$ and $g\in C_b((0,T]\times\mathbb{R}^d)$.

Claim: The Cauchy problem (7.1) has at most one solution $u\in C_b([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T]\times\mathbb{R}^d)$.

Proof: Let $u,v\in C_b([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T]\times\mathbb{R}^d)$ be two solutions of (7.1). Define $w:=u-v$. Then $w\in C_b([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T]\times\mathbb{R}^d)$ and it satisfies

$$D_tw(t,x)-\mathcal{A}w(t,x)=0\leq0\qquad\text{and}\qquad w(0,\cdot)=0\leq0$$

on $(0,T]\times\mathbb{R}^d$ and $\mathbb{R}^d$, respectively. Part (c) now implies $w\leq0$ on $[0,T]\times\mathbb{R}^d$. Repeating the same argument with $-w$ instead of w yields $-w\leq0$ on $[0,T]\times\mathbb{R}^d$. So we have $w=0$ and thus $u=v$ on $[0,T]\times\mathbb{R}^d$.

Exercise 8.4.3

Assumption: Let $\mathcal{A}$ be the elliptic operator from Lecture 7 and let c_0 denote the supremum of the function c over $[0,T]\times\mathbb{R}^d$. Assume that $u\in C_b([0,T]\times\mathbb{R}^d)\cap C^{1,2}((0,T]\times\mathbb{R}^d)$ satisfies the inequality $D_tu\leq\mathcal{A}u$ on

$(0, T] \times \mathbb{R}^d$.

(a) *Assumption:* We define the function $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ by $v(t, x) := e^{-c_0 t} u(t, x) - \|u(0, \cdot)\|_\infty$ and the operator $\mathcal{A}_0$ by $\mathcal{A}_0 u := \mathcal{A} u - c_0 u$.

Claim: The differential inequality $D_t v - \mathcal{A}_0 v \leq 0$ holds true on $(0, T] \times \mathbb{R}^d$.

Proof: Due to $c \leq c_0$ we have

$$\begin{aligned} D_t v(t,x) - \mathcal{A}_0 v(t,x) &= -c_0 e^{-c_0 t} u(t,x) + e^{-c_0 t} D_t u(t,x) - e^{-c_0 t} \mathcal{A} u(t,x) + e^{-c_0 t} c_0 u(t,x) \\ &\quad + (c - c_0)\|u(0,\cdot)\|_\infty \\ &\leq (c_0 - c_0) e^{-c_0 t} u(t,x) + e^{-c_0 t}(D_t u(t,x) - \mathcal{A} u(t,x)) \\ &\leq 0 \end{aligned}$$

on $(0, T] \times \mathbb{R}^d$.

(b) *Claim:* We have $u(t, x) \leq e^{c_0 t}\|u(0, \cdot)\|_\infty$ on $[0, T] \times \mathbb{R}^d$.

Proof: For the function v from part (a) we have $v \in C([0, T] \times \mathbb{R}^d) \cap C^{1,2}((0, T] \times \mathbb{R}^d)$ and

$$|v(t,x)| \leq e^{-c_0 t}|u(t,x)| + \|u(0,\cdot)\|_\infty \leq \max\{1, e^{-c_0 T}\}\|u\|_\infty$$

on $[0, T] \times \mathbb{R}^d$, so that $v \in C_b([0, T] \times \mathbb{R}^d) \cap C^{1,2}((0, T] \times \mathbb{R}^d)$. Furthermore, we compute

$$v(0,\cdot) = u(0,\cdot) - \|u(0,\cdot)\|_\infty \leq 0$$

on $\mathbb{R}^d$. Applying part (c) of Exercise 8.4.2 (with $\mathcal{A}_0$ instead of $\mathcal{A}$) to v thus yields $v \leq 0$, which means $u(t, x) \leq e^{c_0 t}\|u(0, \cdot)\|_\infty$ on $[0, T] \times \mathbb{R}^d$.

(c) *Claim:* If $D_t u = \mathcal{A} u$ on $(0, T] \times \mathbb{R}^d$, then $\|u(t, \cdot)\|_\infty \leq e^{c_0 t}\|u(0, \cdot)\|_\infty$ for every $t \in [0, T]$.

Proof: Due to $D_t u \geq \mathcal{A} u$ we can apply part (b) to $-u$, obtaining $-u(t, x) \geq e^{c_0 t}\|u(0, \cdot)\|_\infty$ on $[0, T] \times \mathbb{R}^d$. Combined with the statement from part (b) the assertion follows.

Exercise 8.4.4

Assumption: Let $f \in C_b(\mathbb{R}^2)$ and $g \in C_b([0, T] \times \mathbb{R}^2)$.

Claim: There exists at most one solution $u \in C_b([0, T] \times \mathbb{R}^2)$ to the Cauchy problem $D_t u = \Delta u + g$ on $(0, T] \times \mathbb{R}^2$ with initial condition $u(0, \cdot) = f$ on $\mathbb{R}^2$ such that $D_t u$, $D_x u$, $D_y u$, $D_{xx} u$ and $D_{yy} u$ are bounded and continuous on $(0, T] \times \mathbb{R}^2$ and $D_{xy} u$ is continuous on $(0, T] \times (\mathbb{R}^2 \setminus \{(0, 0)\})$.

Remark: The function g did not appear in the equation in the original problem, probably by mistake.

Proof: The claim is contained in the statement of the following Lemma, for which we define

$$X := \big\{u \in C^{1,1}((0,T] \times \mathbb{R}^2) \mid \exists\ D_{xx}u,\ D_{yy}u \in C((0,T] \times \mathbb{R}^2),\ \exists\ D_{xy}u,\ D_{yx}u \in C((0,T] \times \mathbb{R}^2 \setminus \{(0,0)\})\big\}.$$

Lemma

For $u \in C_b([0, T], \mathbb{R}^2) \cap X$ the following holds:

(a) If $\partial_t u - \Delta u \leq 0$ on $(0, T] \times \mathbb{R}^2$ and $u(0, \cdot) \leq 0$ on $\mathbb{R}^2$, then $u \leq 0$ on $[0, T] \times \mathbb{R}^2$.

(b) If $\partial_t u - \Delta u = 0$, then

$$\sup_{t \in [0,T]} \|u(t,\cdot)\|_\infty \leq \|u(0,\cdot)\|_\infty.$$

(c) Let $f \in C_b(\mathbb{R}^2)$ and $g \in C_b((0,T] \times \mathbb{R}^2)$, then the Cauchy problem

$$\begin{cases} \partial_t u = \Delta u + g & \text{in } (0,T] \times \mathbb{R}^2, \\ u(0,\cdot) = f & \text{in } \mathbb{R}^2, \end{cases} \tag{1}$$

has at most one solution in $C_b([0,T],\mathbb{R}^2) \cap X$.

(d) There exist $f \in C_b(\mathbb{R}^2)$ and $g \in C_b((0,T] \times \mathbb{R}^2)$ such that the Cauchy problem (1) has no classical solution $u \in C^{1,2}((0,T] \times \mathbb{R}^2) \cap C_b([0,T] \times \mathbb{R}^2)$.

Proof. (a) By Exercise 8.4.2 (a) there exists a $\lambda > 0$ such that $\Delta\varphi \leq \lambda\varphi$ on $\mathbb{R}^2$. Following Exercise 8.4.2 (b), we define for all $n \in \mathbb{N}$ the functions $v_n \in C([0,T] \times \mathbb{R}^2) \cap X$ by

$$v_n(t,x) := u(t,x) - e^{(\lambda+1)t}\tfrac{1}{n}\varphi(x), \quad (t,x) \in [0,T] \times \mathbb{R}^2,$$

which attain each a maximum on $[0,T] \times \mathbb{R}^2$ ans satisfy $v_n(0,\cdot) \leq 0$ on $\mathbb{R}^2$. We have on $(0,T] \times \mathbb{R}^2$ that

$$\begin{aligned} D_t v_n - \Delta v_n &= D_t u - \Delta u - (\lambda+1)\tfrac{1}{n}\varphi e^{(\lambda+1)t} + e^{(\lambda+1)t}\tfrac{1}{n}\Delta\varphi \\ &\leq e^{(\lambda+1)t}(-\lambda\tfrac{1}{n}\varphi + \tfrac{1}{n}\Delta\varphi - \tfrac{1}{n}\varphi) \\ &\leq -e^{(\lambda+1)t}\tfrac{1}{n}\varphi < 0. \end{aligned}$$

Assume (compare with Exercise 8.4.2 (c)) that there exists a $(t_0,x_0) \in (0,T] \times \mathbb{R}^2$ such that $v_n(t_0,x_0) = \max_{(t,x)\in[0,T]\times\mathbb{R}^2} v_n(t,x)$. Then we have $D_t v_n(t_0,x_0) \geq 0$ (since $t_0 = T$ is possible). Moreover, the function $s \mapsto v_n(t_0, x_0 + se_j)$ attains a maximum at $s = 0$. Therefore, the terms $D_{jj}v_n(t_0,x_0)$ are nonpositive for $j = 1,2$. Hence,

$$0 > D_t v_n(t_0,x_0) - \Delta v_n(t_0,x_0) = D_t v_n(t_0,x_0) - \sum_{j=1}^{2} D_{jj} v_n(t_0,x_0) \geq 0.$$

By this contradiction we conclude $\max_{[0,T]\times\mathbb{R}^2} v_n = \max_{\mathbb{R}^2} v_n(0,\cdot) \leq 0$. The pointwise limits yields $u \leq 0$ on $[0,T] \times \mathbb{R}^2$.

(b) Following the proof of Exercise 8.4.3 we set $v(t,x) := u(t,x) - \|u(0,\cdot)\|_\infty$, $(t,x) \in [0,T] \times \mathbb{R}^2$ and have $D_t v - \Delta v = 0$ on $(0,T] \times \mathbb{R}^2$. Hence, (a) gives $v \leq 0$ on $[0,T] \times \mathbb{R}^2$ and consequently we have $u(t,\cdot) \leq \|u(0,\cdot)\|_\infty$ on $\mathbb{R}^d$ for all $t \in [0,T]$. With the function $\tilde{v}(t,x) := -u(t,x) - \|u(0,\cdot)\|_\infty$ we get in the same way $u(t,\cdot) \geq -\|u(0,\cdot)\|_\infty$ on $\mathbb{R}^d$ for all $t \in [0,T]$, so that together we have $\|u(t,\cdot)\|_\infty \leq \|u(0,\cdot)\|_\infty$ on $\mathbb{R}^d$ for all $t \in [0,T]$.

(c) Let $f \in C_b(\mathbb{R}^2)$ and $g \in C_b((0,T] \times \mathbb{R}^2)$ and $u_1, u_2 \in C_b([0,T] \times \mathbb{R}^2) \cap X$ solutions of the Cauchy problem (1). Then we have for $u := u_1 - u_2$ that $u(0,\cdot) = 0$ on $\mathbb{R}^2$ and $D_t u - \Delta u = 0$ on $(0,T] \times \mathbb{R}^2$. Thus, by (b), we conclude $\|u(t,x)\|_\infty \leq \|u(0,\cdot)\|_\infty = 0$ for all $t \in [0,T]$, so that $u_1 = u_2$ on $[0,T] \times \mathbb{R}^2$.

(d) We define

$$\tilde{u}(t,x) := \sum_{n=1}^{\infty} \tfrac{1}{n} xy\eta(2^n x, 2^n y), \ (x,y) \in \mathbb{R}^2$$

where $\eta \in C_c^\infty(\mathbb{R}^2)$ with $\chi_{B(0,1)} \leq \eta \leq \chi_{B(0,2)}$. Then by Example 8.2.1 in Lecture 8 we have $\tilde{u} \in C_b^1(\mathbb{R}^2)$ with $D_{xx}\tilde{u}, D_{yy}\tilde{u} \in C_b(\mathbb{R}^2)$ and $D_{xy}\tilde{u} \in C(\mathbb{R}^2 \backslash \{(0,0)\})$, where $|D_{xy}\tilde{u}(w,z)| \to \infty$ for $(w,z) \to (0,0)$. However, $\tilde{u} \in C_b([0,T] \times \mathbb{R}^2) \cap X$ and

$$D_t\tilde{u} = 0 = \Delta\tilde{u} - \Delta\tilde{u} =: \Delta\tilde{u} + g \ \text{ in } (0,T] \times \mathbb{R}^2.$$

Thus, by (c), the Cauchy problem (1) for $g = -\Delta\tilde{u}$ and $f = \tilde{u}(0,\cdot)$ has no classical solution $u \in C_b([0,T] \times \mathbb{R}^2) \cap C^{1,2}((0,T] \times \mathbb{R}^2) \subset C_b([0,T] \times \mathbb{R}^2) \cap X$. □